$$Gb(p,q,q) \le s[Gb(p,p,p) + kGb(p,q,q)]$$
$$= skGb(p,q,q).$$

It follows that, $(1-sk)Gb(p,q,q)<0 \Rightarrow Gb(p,q,q)=0$; since $sk\in[0,1)$. To show that T is Gb-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} \{y_n\} = p$. Consider

$$Gb(p,Ty_n,Ty_n) \le Gb(Tp,Ty_n,Ty_n)$$
$$\le k\varphi(Gb(p,y_n,y_n))$$
$$\le kGb(p,y_n,y_n).$$

As $n \to \infty, y_n \to p$, we get,

$$Gb(p,Ty_n,Ty_n) \le kGb(p,p,p)=0.$$

Thus $Ty_n=p=Tp$. It is proved that T is Gb-continuous at p.

Corollary 3.3. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying for some $m\in N$

$$Gb(T^m x,T^m y,T^m z) \le k\varphi(Gb(x,y,z)); \qquad (3.4)$$

for all $x,y,z\in X, sk\in[0,1)$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T^m is Gb-continuous at p.

Proof: Here $T(u)=T(T^m u)=T^{(m+1)} u=T^m (Tu)$. Therefore by Theorem (3.2) we conclude that T^m has a fixed point say p. Also we have Tu a fixed point to T^m. So $Tu = u$, and T has unique fixed point.

Corollary 3.4. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying for some $m\in N$

$$Gb(Tx,Ty,Ty) \le k\varphi(Gb(x,y,y)); \qquad (3.5)$$

for all $x,y,z\in X, sk\in[0,1)$. Then T has a unique fixed point (say u, i.e., $Tu=u$), and T is Gb-continuous at p.

Proof: Taking $z = y$ in Theorem (3.2).

Corollary 3.5. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying for some $m\in N$,

$$Gb(Tx,Ty,Tz) \le kGb(x,y,z); \qquad (3.6)$$

for all $x,y,z\in X, sk\in[0,1)$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is Gb-continuous at p.

Proof: To prove this corollary we define the φ function as $\varphi:[0,+\infty]\to[0,+\infty]$ be a nondecreasing function with $\lim_{n,m\to\infty} \varphi^n (t)=0$ for all $t\in(0,+\infty)$ and $\varphi(\omega)=\omega$. Clearly φ is non-decreasing function with $\lim_{n,m\to\infty} \varphi^n (t)=0$ for all $t\in(0,+\infty)$).

Since $Gb(Tx,Ty,Tz) \le k\varphi(Gb(x,y,z))$; for all $x,y,z\in X, sk\in[0,1$. Therefore by Theorem (3.2) we get required result.

Corollary 3.6. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying for some $m\in N$

$$Gb(Tx,Ty,Tz) \le (Gb(x,y,z))/(1+Gb(x,y,z)), \qquad (3.7)$$

for all $x,y,z\in X$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is Gb-continuous at p.

Proof: To prove this corollary we define the φ function as $\varphi:[0,+\infty]\to[0,+\infty]$ be a non-decreasing function with $\lim_{n,m\to\infty} \varphi^n (t)=0$ for all $t\in(0,+\infty)$ and $\varphi(w)=kw/(1+kw)$. Clearly φ be a non-decreasing function with $\lim_{n,m\to\infty} \varphi^n (t)=0$ for all $t\in(0,+\infty)$.

Theorem 3.7. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying

$$Gb(Tx,Ty,Tz) \le k\varphi \max\{Gb(x,y,z),Gb(x,Tx,Tx),Gb(y,Ty,Ty),Gb(z,Tz,Tz) \qquad (3.8)$$

for all $x,y,z\in X, sk\in[0,1)$. Then T has a unique fixed point (say p, i.e., $Tp=p$), and T is Gb-continuous at p.

Proof: Similar as theorem 3.2

Corollary 3.8. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying for some $m\in N$

$$Gb(Tx,Ty,Tz) \le k \max\{Gb(x,y,z),Gb(x,Tx,Tx),Gb(y,Ty,Ty),Gb(z,Tz,Tz)\} \qquad (3.9)$$

for all $x,y,z\in X, sk\in[0,1)$. Then T has a unique fixed point (say u, i.e., $Tu=u$), and T is Gb-continuous at p.

Corollary 3.9. Let (X,Gb) be a complete Gb-metric space and let $T:X\to X$ be a mapping satisfying for some $m\in N$

$$Gb(Tx,Ty,Tz) \le (Gb(x,y,z))/(1+Gb(x,y,z)),$$

where $M(x,y,z)=k\max\{Gb(x,y,z),Gb(x,Tx,Tx),Gb(y,Ty,Ty)\}$ (3.10) for all $x,y,z\in X$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is Gb-continuous at p.

Proof: To prove this corollary we define the φ function as $\varphi:[0,+\infty]\to[0,+\infty]$ be a non-decreasing function with $\lim_{n,m\to\infty} \varphi^n (t)=0$ for all $t\in(0,+\infty)$ and $\varphi(w)=w/(1+w)$. Clearly φ be a non-decreasing function.

Example 3.10. Let us define $Gb(x,y,z)=|x-y|+|y-z|+|x-z|$ and let $x\in X$. Let $T(x)=x/3$. Without loss of generality, we assume $x>y>z$ and $\varphi(t)=t$. Then (X,Gb) be a complete Gb-metric space.

Then (i)

$$Gb(Tx,Ty,Tz)=|x/3-y/3|+|y/3-z/3|+|x/3-z/3|$$
$$= 1/3 \{|x-y|+|y-z|+|x-z|\}$$
$$\le 1/2 \{|x-y|+|y-z|+|x-z| \}$$
$$\le k(Gb(x,y,z))$$
$$= k\varphi(Gb(x,y,z)).$$

(ii) $Gb(Tx,Ty,Tz)=|x/3-y/3|+|y/3-z/3|+|x/3-z/3| \le 2k|x-x/3|$

$$\le k\varphi \max\{Gb(x,y,z),Gb(x,Tx,Tx),Gb(y,Ty,Ty),Gb(z,Tz,Tz) \}$$

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Fixed Point Theorems Satisfying Φ- Maps in Gb-Metric Space

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ABSTRACT

In this paper, we prove fixed point theorems for self mapping $T:X\to X$ in a complete Gb-metric space for a Φ-maps as $\varphi:[0,+\infty]\to[0,+\infty]$ be a non-decreasing map with $\lim_{n,m\to\infty} \varphi^n(t)=0$ for all $t\in(0,+\infty)$ and also prove uniqueness for such fixed points in respective contractions. Our results are supported by an example.

INTRODUCTION AND PRELIMINARIES

The fixed point theory which plays very important role in applied mathematics and sciences. So the metric spaces are generalized by many authors by various ways. Czerwik [6] introduced b-metric space. Zead Mustafa and Brailey Sims [11] coined the concept of G-metric space. A. Aghajani, M. Abbas and J. R. Roshan [2] extended the G-metric space with b-metric space and develop the new structure of metric space, which is generalized metric space called Gb-metric space. In this paper for a self mapping in a Gb metric space we study some fixed point theorems under some contractions [15], [10]-[9] related to a non-decreasing map [4] $\varphi:[0,+\infty]\to[0,+\infty]$ with $\lim_{n,m\to\infty} \varphi^n(t)=0$ for all $t\in(0,+\infty)$.

BASIC CONCEPTS

A b-metric space is defined by Czerwik [6] as follows.

Definition 2.1. [6] Let X be a non empty set and the mapping $d:X\times X\to[0,\infty)$. The mapping d satisfies

i) $d(x,y)=0$ if and only if $x=y$ for all $x,y\in X$,

ii) $d(x,y)=d(y,x)$ for all $x,y\in X$,

iii) there exists a real number $s\geq 1$ such that $d(x,y)\leq s[d(x,z)+d(z,y)]$ for all $x,y,z\in X$. Then d is called a b-metric on X. The ordered pair (X, d) is called b-metric space with coefficient s.

Definition 2.2. [11] Let X be a non empty set and the mapping $G:X\times X\times X\to[0,\infty)$.

The mapping G satisfies

i) $G(x,y,z)=0$ if and only if $x=y=z$ for all $x,y,z\in X$,

ii) $0<G(x,x,y)$ for all $x,y\in X$,

iii) $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y$,

Key words and phrases. G-metric spaces; b-metric spaces; Gb-metric spaces; contraction mappings.

iv) $G(x,y,z)=G(x,z,y)=G(y,z,x)=$(symmetry in all three variables),

v) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$ (rectangle inequality). Then G is called a G-metric on X and (X, G) is called G-metric space.

Aghajani and et.al [2] defined Gb-metric space as follows

Definition 2.3. [2] Let X be a nonempty set and $s\geq 1$ be a given real number. Suppose that a mapping $Gb:X\times X\times X\to R+$ satisfies:

i) $Gb(x,y,z)=0$ if $x=y=z$ for all $x,y,z\in X$,

ii) $0<Gb(x,x,y)$ for all $x,y,z\in X$ with $x\neq y$,

iii) $Gb(x,x,y)\leq Gb(x,y,z)$ for all $x,y,z\in X$ with $y\neq z$

iv) $Gb(x,y,z)=Gb(px,z,y)$, where p is a permutation of x,y,z (symmetry),

v) $Gb(x,y,z)\leq s[Gb(x,a,a)+Gb(a,y,z)]$.

Then Gb is called a generalized b-metric or Gb-metric on X. The ordered pair (X,Gb) is called generalized b-metric or Gb-metric space.

Following example shows that a Gb-metric on X need not be a G-metric on X.

Example 2.4. [2] Let (X,G) be a G-metric space and $G^*(x,y,z)=[G(x,y,z)]^p$; where $p>1$ is a real number. Note that $G*$ is a Gb-metric with $s=2^{(P-1)}$. Obviously, $G*$ satisfies conditions to (iv) of the Gb-metric space, so it suffices to show that condition (v) of Gb-metric space is hold. If $1<p<\infty$, then the convexity of the function $f(x)=x^p$ $(x>0)$ implies that $(a+b)^p\leq 2^{(P-1)}(a^p+b^p)$. Thus for each $x,y,z,a\in X$ we obtain

$$G^*(x,y,z)=[G(x,y,z)]^p\leq[(G(x,a,a)+G(a,y,z))]^p$$
$$\leq 2^{(P-1)}([G(x,a,a)]^p+[G(a,y,z)]^p)$$
$$=2^{(P-1)}(G^*(x,a,a)+G^*(a,y,z)).$$

So $G*$ is a Gb-metric with $s=2^{(P-1)}$.

Also in the above example, $(X, G*)$ is not necessarily a G-metric space.

Example 2.5. Let $X=R$ and let $Gb(x,y,z)=\max\{|x-y|^2,|y-z|^2,|z-x|^2\}$.

Then (X,Gb) is a Gb-metric space with the coefficient $s=2$.

Definition 2.6. [2] Let X be a Gb-metric space. A sequence $\{x_n\}$ in X is said to be:
i) Gb-Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer n0 such that for all $m,n,l \geq n_0, G(x_n,x_m,x_l)<\epsilon$;
ii) Gb-convergent to a point $x\in X$ if, for each $\epsilon>0$, there exists a positive integer n0 such that for all $m,n\geq n_0$, $G(x_n,x_m,x)<\epsilon$.

Proposition 2.7. [2] Let (X,Gb) be a Gb-metric space. Then the following are equivalents:
i) $\{x_n\}$ is Gb-convergent to x.
ii) $G(x_n,x_n,x)\to 0$, as $n\to\infty$.
iii) $G(x_n,x,x) \to 0$, as $n\to\infty$.

Proposition 2.8. [2] Let (X,Gb) be a Gb-metric space. Then the following are equivalents:
i) The sequence $\{x_n\}$ is Gb-Cauchy.
ii) For every $\epsilon>0$, there exists $n_0\in N$ such that $G(x_n,x_m,x_m)<\epsilon$, for all $n,m\geq n_0$.

Definition 2.9. [2] A Gb-metric space X is called Gb-complete if every Gb-Cauchy sequence is Gb-convergent in X.

MAIN RESULTS

Our first main result is

Definition 3.1. [4] Let Φ be the set all functions φ such that $\varphi: [0,+\infty]\to[0,+\infty]$ be a non-decreasing function with
i) $\lim_{(n,m\to\infty)} [\varphi^n (t)]=0$ for all $t\in(0,+\infty)$,
ii) $\varphi(t)\leq t$ for all $t\in(0,+\infty)$,
iii) $\varphi(0)=0$.
Then $\varphi \in \Phi$, φ is called Φ-maps.

Theorem 3.2. Let (X, Gb) be a complete Gb-metric space with and let $T:X\to X$ be a mapping satisfying

$$Gb(Tx,Ty,Tz)\leq k\varphi(Gb(x,y,z)) \quad (3.1)$$

for all $x,y,z\in X,\varphi\in\Phi,sk\in[0,1)$. Then T has a unique fixed point (say p, i.e., Tp = p), and T is Gb-continuous at p.

Proof: Let $x_0\in X$ and the mapping $T:X\to X$ be a self map. Then, we get a sequence $\{x_n\}$ in X such that $x_n=Tx_{(n-1)}=T^n x_0$. If $x_n=x_{(n-1)}$ for each $n\in N$. Then clearly $\{x_n\}$ is Gb-Cauchy sequence. Suppose $x_n\neq x_{(n-1)}$ for each $n\in N$. We claim that $\{x_n\}$is a Gb-Cauchy sequence in X, for $n\in N$. Consider for $n\in N$,

$$\begin{aligned} Gb(x_n,x_{(n+1)},x_{(n+1)}) &= Gb([Tx]_{(n-1)},[Tx]_n,[Tx]_n) \\ &\leq k\varphi(Gb(x_{(n-1)},x_n,x_n)) \\ &\leq kGb(x_{(n-1)},x_n,x_n) \\ &\leq k^2\, Gb(x_{(n-2)},x_{(n-1)},x_{(n-1)}) \\ &\leq\cdots\leq k^n\, \varphi(Gb(x_0,x_1,x_1)) \\ &\leq k^n\, Gb(x_0,x_1,x_1)) \end{aligned}$$

For given $\epsilon > 0,\varphi(\epsilon) \leq \epsilon$ there is an integer n_0 such that

$$Gb(x_n,x_{(n+1)},x_{(n+1)})<\epsilon/s-k\varphi(\epsilon),sk \in [0,1),n \geq n_0 \quad (3.2)$$

For $n,m\in N,n<m$, we claim that $Gb(x_n,x_m,x_m)<\epsilon$, (3.3) for all $m,n \geq n_0$. We prove inequality (3.3) by induction on m, by equation (3.2) the inequality (3.3) hold for m=n+1. Assume that inequality (3.3) hold for m=k, therefore $Gb(x_n,x_k,x_k)<\epsilon$. Consider $m = k + 1$,

$$\begin{aligned} Gb(x_n,x_m,x_m) &= Gb(x_n,x_{(k+1)},x_{(k+1)}) \\ &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)}) + Gb(x_{(n+1)},x_{(k+1)},x_{(k+1)})] \\ &= s[Gb(x_n,x_{(n+1)},x_{(n+1)}) + Gb(Tx_n,[Tx]_k,[Tx]_k)] \\ &\leq s\, [Gb(x_n,x_{(n+1)},x_{(n+1)})+ k\varphi(Gb(x_n,x_k,x_k))] \\ &< s[\epsilon/s-k\varphi(\epsilon)+k\varphi(\epsilon)] \\ &= \epsilon \end{aligned}$$

Therefore, by induction on m the inequality (3.3) hold for all $n\geq m\geq n_0$. Hence $\{x_n\}$ is a Gb-Cauchy sequence in X. By Gb-completeness of X, there exists $p\in X$ such that $\{x_n\}$ is Gb-converges to p. Now we show that p is fixed point of T. Suppose that $T(p)\neq p$.

$$\begin{aligned} Gb(x_n,Tp,Tp) &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)})+Gb(x_{(n+1)},Tp,Tp)] \\ &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)}) + k\varphi(Gb(x_n,p,p))] \\ &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)}) + k(Gb(x_n,p,p))]. \end{aligned}$$

As $n\to+\infty,x_n\to p,Gb(p,Tp,Tp)\leq 0$ and since $Gb(p,Tp,Tp)\geq 0$. Then Tp=p.
This is contradiction to $Tp \neq p$. Therefore p is a fixed point of T. For uniqueness suppose $q \,/= p$ and q is another fixed point of T, $Tq = q$.

$$\begin{aligned} Gb(x_n,Tq,Tq) &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)})+Gb(x_{(n+1)},Tq,Tq)] \\ &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)}) + k\varphi(Gb(x_n,q,q))] \\ &\leq s[Gb(x_n,x_{(n+1)},x_{(n+1)}) + k(Gb(x_n,q,q))]. \end{aligned}$$

As $n \to \infty,x_n\to p$ and $Tq = q$, we get,

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