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Some Fixed Point Theorems in b_G-Metric Space

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Abstract:
A new metric which is derived from two metrics called G-metric and b-metric which were introduced by Mustafa and Sims [8] and Cezerwik [3], respectively. As an application of this newly developed space, we have established some fixed Point theorems using this metric space.

I. INTRODUCTION AND PRELIMINARIES

The metric space has generalized by many ways. Czerwik [3] introduced the concept of b-metric space. We have noted various types of fixed point's results in [1, 5, 2, 4, 6]. Zead Mustafa and Brailey Sims [8] poised this and introduced the concept of G-metric space and proved some fixed point theorems. We have observed lot of fixed point in G-metric spaces in [7, 8, 9, 10, 11, 12, 13] from In this paper, we extend the G-metric space and develop the new structure of metric space, which we call b_G-metric space. Initially Cezerwik [3] defined b-metric space which follows:

Definition 1.1 Let X be a non empty set and the mapping $d: X \times X \to [0, \infty)$. The mapping d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there exists a real numbers $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a b-metric on X and the ordered pair (X, d) is called b-metric space with coefficients$s$.

For$s = 1$, the b-metric coincide with usual metric space. Z. Mustafa and B. Sim [8] introduced G-metric space as follows:

Definition 1.2 Let X be a non empty set and the mapping $G: X \times X \times X \to [0, \infty)$. The mapping G satisfies
$G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$;

(i) $0 < G(x, x, y)$ for all $x, y \in X$;
(ii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iii) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);
(iv) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a G-metric on X and the pair (X, G) is called G-metric space. We combined the above two generalized metric spaces and defined the b_G-metric space as follows:

Definition 1.3 Let X be a non empty set and the mapping $b_G: X \times X \times X \to [0, \infty)$. The mapping b_G satisfies

(i) $b_G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$;
(ii) $b_G(x, x, y) \leq b_G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iii) $b_G(x, y, z) = b_G(x, z, y) = b_G(y, z, x) = \cdots$ (symmetry in all three variables);
(iv) there exists a real numbers $s \geq 1$ such that
$$b_G(x, y, z) \leq s[b_G(x, a, a) + b_G(a, y, z)],$$
for all $x, y, z, a \in X$.

Then b_G is called a b_G metric on X. The ordered pair (X, b_G) is called b_G-metric space.

Example 1.4 Let $X = R$ and let
$$b_G(x, y, z) = \max\{|x - y|^2, |y - z|^2, |x - z|^2\}.$$
Then (X, b_G) is a b_G-metric space with the coefficient $s = 2$.

Example 1.5 Let $X = R^+$, $p > 1$ a constant and $b_G: X \times X \times X \to [0, \infty)$ be defined by
$$b_G(x, y, z) = \max\{(4x, y, z)^p, |4x - y - z|^p\},$$
for all $x, y, z \in X$. Then (X, b_G) is a b_G-metric space with$s > 1$.

Definition 1.6 Let (X, b_G) be a b_G-metric space, and let (x_n) be sequence of points in X, a point $x \in X$ is said to be the limit of the sequence (x_n), if $lim_{n,m\to\infty} b_G(x, x_n, x_m) = 0$.
In other words, if $x_n \to x$ in a b_G-metric space(X, b_G), it mean that for any given$\varepsilon > 0$, there exists $N \in N$ such that $b_G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. In this case we say that the sequence (x_n) is b_G-converges to x.

Definition 1.7 Let (X, b_G) be a b_G- metric space, and a sequence (x_n) is called b_G-Cauchy, if, $\lim_{m,n,l\to\infty} b_G(x_n, y_m, z_l) = 0$ for all $x, y \in X$. Thus, that if $x_n \to x$ in a b_G-metric space (X, b_G), if for any $\varepsilon > 0$, there exists $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for $n, m, l \geq N$.

Definition 1.8 Let (X, b_G) be a b_G- metric space. It is said to be complete if every b_G-Cauchy sequence converges in X.

Proposition 1.9 Let (X, b_G) be a b_G-metric space, then the following are equivalents:
(x_n) is b_G-convergent to x.

(i) $G(x_n, x_m, x) \to 0$, as $n \to \infty$.
(ii) $G(x_m, x, x) \to 0$, as $n \to \infty$.
(iii) $G(x_m, x_n, x) \to 0$, as $n \to \infty$.

Proposition 1.10 *Let* (X, b_G) be a b_G-metric space, then the following are equivalents.

(i) The sequence (x_n) is b_G-Cauchy.

(ii) For every$\varepsilon > 0$, there exists $N \in N$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$.

II. MAIN RESULTS

The first result of this paper is

Theorem 2.1 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying

$$G(Tx, Ty, Tz) \leq kG(x, y, z);$$

(2.1)

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence (x_n) in X by $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq kG(x_{n-1}, x_n, x_n)$$
$$\leq k^2 G(x_{n-2}, x_{n-1}, x_{n-1})$$
$$\leq k^n G(x_0, x_1, x_1).$$

Moreover, for all $n, m \in N, n < m$ and using property (iv) of definition 1.3, we have

$$G(x_n, x_m, x_m) \leq$$
$$s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$$
$$\leq s[k^n G(x_0, x_1, x_1) + G(x_{n+1}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3[G(x_{n+2}, x_{n+3}, x_{n+3}) + G(x_{n+3}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3 k^{n+2} G(x_0, x_1, x_1) + s^3 G(x_{n+3}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3 k^{n+2} G(x_0, x_1, x_1) + \ldots. + s^{m-1} k^{n+m-2} G(x_0, x_1, x_1) + s^{m-1} k^{n+m-1} G(x_0, x_1, x_1)$$
$$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3 + \cdots + (sk)^{m-2}) + (sk)^{m-2} k] G(x_0, x_1, x_1)$$
$$\leq sk^n\left[\frac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2} k\right] G(x_0, x_1, x_1).$$

Since $k \in (0,1)$. Letting $m, n \to \infty$, $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. Hence (x_n) is a Cauchy sequence in X. Since X is b_G complete, there exists $u \in X$ such that the sequence (x_n) is b_G-converges to u. Now we claim that u is fixed point of T i.e. $u = Tu$. Suppose $T(u) \neq u$. Consider

$$G(x_n, Tu, Tu) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tu, Tu]$$
$$\leq s[k^n G(x_0, x_1, x_1) + G(x_n, u, u)].$$

As $n \to \infty, x_n \to u$, we have

$$G(u, Tu, Tu) \leq sG(u, u, u).$$

This shows that $G(u, Tu, Tu)$ and hence$u = Tu$. This is contradiction to $T(u) = u$. Therefore u is a fixed point of$T$. Suppose $v \neq u$ and v is another fixed point of T i.e. $Tv = v$.

$$G(x_n, Tv, Tv) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tv, Tv)]$$
$$G(x_n, v, v) \leq s[k^n G(x_0, x_1, x_1) + G(x_n, v, v)].$$

As $n \to \infty, x_n \to u$ and

$$G(u, v, v) \leq sG(u, v, v)].$$

Since $s \geq 1$, we must have $G(u, v, v) = 0$. Hence $u = v$. Now we will show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$$G(u, Ty_n, Ty_n) \leq kG(u, y_n, y_n).$$

But

$$G(y_n, Ty_n, Ty_n) \leq s[G(y_n, u, u) + G(u, Ty_n, Ty_n)]$$
$$\leq s[G(y_n, u, u) + kG(u, y_n, y_n)].$$

As $n \to \infty$, we have

$$G(u, Ty_n, Ty_n) \leq s[G(u, u, u) + kG(u, u, u)]$$
$$G(u, Ty_n, Ty_n) = 0.$$

Hence $Ty_n = u = Tu$. This show that T is b_G-continuous at u.

Corollary 2.2 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$$G(T^m(x), T^m(y), T^m(z)) \leq kG(x, y, z);$$

(2.2)

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u, i.e. $Tu = u$) and T^m is b_G-continuous at u.

Theorem 2.3 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to$ X be a mapping satisfying

$$G(Tx, Ty, Tz) \leq k[G(x, Tx, Tx) + G(y, Ty, Ty)];$$

(2.3)

for all $x, y, z \in X$ and $k \in [0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence (x_n) in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq k[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})]$$

$\leq kG(x_{n-1},x_n,x_n)+kG(x_n,x_{n+1},x_{n+1})$

$\leq \frac{k}{1-k}G(x_{n-1},x_n,x_n)$

$\leq \lambda G(x_{n-1},x_n,x_n)$

$\leq \lambda^n G(x_0,x_1,x_1)$,

where$\lambda=\frac{k}{1-k}<1$, since $k\in[0,1/2)$. Moreover, for all $n,m\in N, n<m$ and by property (iv) in definition 1.3, we have

$G(x_n,x_m,x_m)\leq$

$s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_m,x_m)]$

$\leq s[\lambda^n G(x_0,x_1,x_1)+G(x_{n+1},x_m,x_m)]$

$\leq s\lambda^n G(x_0,x_1,x_1)+s^2[G(x_{n+1},x_{n+2},x_{n+2})+$
$G(x_{n+2},x_m,x_m)]$

$\leq s\lambda^n G(x_0,x_1,x_1)+s^2\lambda^{n+1}G(x_0,x_1,x_1)+$
$s^3[G(x_{n+2},x_{n+3},x_{n+3})+G(x_{n+3},x_m,x_m)]$

$\leq s\lambda^n G(x_0,x_1,x_1)+s^2\lambda^{n+1}G(x_0,x_1,x_1)+$
$s^3\lambda^{n+2}G(x_0,x_1,x_1)+s^3G(x_{n+3},x_m,x_m)]$

$\leq s\lambda^n G(x_0,x_1,x_1)+s^2\lambda^{n+1}G(x_0,x_1,x_1)+$
$s^3\lambda^{n+2}G(x_0,x_1,x_1)+\ldots.+s^{m-1}\lambda^{n+m-2}G(x_0,x_1,$
$x_1)+s^{m-1}\lambda^{n+m-1}G(x_0,x_1,x_1)$

$\leq s\lambda^n[(1+s\lambda+(s\lambda)^2+(s\lambda)^3+\ldots.+(s\lambda)^{m-2})+$
$(s\lambda)^{m-2}\lambda]G(x_0,x_1,x_1)$

$\leq s\lambda^n[\frac{1-(s\lambda)^{n-(m-2)}}{(1-s\lambda)}+(s\lambda)^{m-2}\lambda]G(x_0,x_1,x_1)$.

Letting $m,n\to\infty$, we have $\lim_{n,m\to\infty}G(x_n,x_m,x_m)=0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u\in X$ such that (x_n) is b_G converges to u. We claim that u is fixed point of T. Consider

$G(x_n,Tu,Tu)\leq s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},Tu,Tu)]$

$\leq s[\lambda^n G(x_0,x_1,x_1)+k[G(x_n,x_{n+1},x_{n+1})+G(u,Tu,Tu)]$

$\leq s\lambda^n(1+k)G(x_0,x_1,x_1)+$
$skG(u,T(u),T(u))$.

As $n\to\infty, x_n\to u$,

$G(u,Tu,Tu)\leq skG(u,Tu,Tu)$.

This implies that $G(u,Tu,Tu)=0$ and hence $Tu=u$ i.e. u is a fixed point of T.

Suppose $v\neq u$ is another fixed point of T i.e. $T(v)=v$. Now

$G(x_n,Tv,Tv)\leq s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},Tv,Tv)]$

$G(x_n,v,v)\leq s[\lambda^n G(x_0,x_1,x_1)+$
$k[G(x_n,x_{n+1},x_{n+1})+G(v,v,v)]$

$\leq s\lambda^n(1+k)G(x_0,x_1,x_1)+skG(v,v,v)$.

As $n\to\infty, x_n\to u$

$G(u,v,v)\leq skG(u,v,v)]$.

Since $s\geq 1$, hence $G(u,v,v)=0$ and $u=v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty}y_n=u$. Consider

$G(u,Ty_n,Ty_n)\leq k[G(u,u,u)+G(y_n,Ty_n,Ty_n)]$
$\leq kG(y_n,Ty_n,Ty_n)$.

But

$G(y_n,Ty_n,Ty_n)\leq s[G(y_n,u,u)+G(u,Ty_n,Ty_n)]$

$\leq s[G(y_n,u,u)+kG(y_n,Ty_n,Ty_n)]$.

As $n\to\infty$,

$G(u,Ty_n,Ty_n)\leq s[G(u,u,u)+kG(u,Ty_n,Ty_n)]$
$\leq skG(u,Ty_n,Ty_n)$.

Since $s\geq 1$ and $k\in[0,1/2)$, we must have $G(u,Ty_n,Ty_n)=0$. Hence $T(y_n)=u=T(u)$. This show that T is b_G-continuous at u.

Corollary 2.4 Let (X,b_G) be a complete b_G-metric space with $s\geq 1$ *and let* $T:X\to X$ be a mapping satisfying for some $m\in N$

$G(T^m(x),T^m(y),T^m(z))\leq k[G(x,T^m(x),T^m(x))+$
$G(y,T^m(y),T^m(y))]$ (2.4)

for all $x,y,z\in X$ and $k\in[0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu=u$) and T^m is b_G-continuous at u.

Theorem 2.5 *Let* (X,b_G) be a complete b_G-metric space with $s\geq 1$ and let $T:X\to X$ be a mapping satisfying

$G(Tx,Ty,Tz)\leq k\max[G(x,y,z),G(x,Tx,Tx)+$
$G(y,Ty,Ty)]$ (2.5)

for all $x,y,z\in X$ and $k\in[0,1)$. Then T has a unique fixed point (say u i.e. $Tu=u$) and T is b_G-continuous at u.

Proof: Let $x_0\in X$. Define a sequence x_n in X such that $x_n=Tx_{n-1}=T^n x_0$. Consider

$G(x_n,x_{n+1},x_{n+1})=G(Tx_{n-1},Tx_n,Tx_n)$

$\leq$
$k\max[G(x_{n-1},x_n,x_n),G(x_{n-1},x_n,x_n),G(x_n,x_{n+1},x_{n+1})]$

$\leq kG(x_{n-1},x_n,x_n)$

$\leq k^n G(x_0,x_1,x_1)$.

Moreover for all $n,m\in N, n<m$ and by property (iv) in definition 1.3, we have

$$G(x_n, x_m, x_m) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$$
$$\leq s[k^n G(x_0, x_1, x_1) + G(x_{n+1}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3[G(x_{n+2}, x_{n+3}, x_{n+3}) + G(x_{n+3}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3 k^{n+2} G(x_0, x_1, x_1) + s^3 G(x_{n+3}, x_m, x_m)]$$
$$\leq sk^n G(x_0, x_1, x_1) + s^2 k^{n+1} G(x_0, x_1, x_1) + s^3 k^{n+2} G(x_0, x_1, x_1) + \ldots + s^{m-1} k^{n+m-2} G(x_0, x_1, x_1) + s^{m-1} k^{n+m-1} G(x_0, x_1, x_1)$$
$$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3 + \ldots + (sk)^{m-2}) + (sk)^{m-2} k] G(x_0, x_1, x_1)$$
$$\leq sk^n\left[\frac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2} k\right] G(x_0, x_1, x_1).$$

Letting $m, n \to \infty$, we have $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that the sequence (x_n) is b_G converges to u. We claim that u is fixed point of T. Consider

$$G(x_n, Tu, Tu) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tu, Tu)]$$
$$\leq s[k^n G(x_0, x_1, x_1) + k\max[G(x_n, u, u), G(x_n, x_{n+1}, x_{n+1}), G(u, Tu, Tu)]$$
$$\leq sk^n G(x_0, x_1, x_1) + skG(u, Tu, Tu).$$

As $n \to \infty, x_n \to u$, we have

$$G(u, Tu, Tu) \leq skG(u, Tu, Tu).$$

Since $s \geq 1$ and $k \in [0,1)$, we must have $G(u, Tu, Tu) = 0$ and hence $Tu = u$.

Suppose $v \neq u$ is another fixed point of T i.e. $Tv = v$. Consider

$$G(x_n, Tv, Tv) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, Tv, Tv)]$$

$$G(x_n, v, v) \leq s[k^n G(x_0, x_1, x_1) + k\max[G(x_n, v, v), G(x_n, x_{n+1}, x_{n+1}), G(v, v, v)]$$
$$\leq sk^n G(x_0, x_1, x_1) + skG(v, v, v).$$

As $n \to \infty, x_n \to u$. Hence $G(u, v, v) = 0$, since $s \geq 1$. Thus $u = v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$$G(u, Ty_n, Ty_n) \leq kmax[G(u, y_n, y_n), G(y_n, Ty_n, Ty_n), G(y_n, Ty_n, Ty_n)]$$
$$\leq kG(y_n, Ty_n, Ty_n).$$

But

$$G(y_n, Ty_n, Ty_n) \leq s[G(y_n, u, u) + G(u, Ty_n, Ty_n)]$$
$$\leq s[G(y_n, u, u) + kG(y_n, Ty_n, Ty_n)].$$

Since $s \geq 1$ and $k \in [0,1)$, applying $n \to \infty$, we have

$$G(u, Ty_n, Ty_n) \leq s[G(u, u, u) + kG(u, Ty_n, Ty_n)]$$
$$\leq skG(u, Ty_n, Ty_n)$$

This implies $G(u, T(y_n), T(y_n)) = 0$. Hence $Ty_n = u = T(u)$. It shows that T is b_G-continuous at u.

Corollary 2.6 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$$G(T^m x, T^m y, T^m z) \leq k\max[G(x, y, z), G(x, T^m x, T^m x), G(y, T^m y, T^m y)]; \quad (2.6)$$

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T^m is b_G-continuous at u.

Theorem 2.7 Let (X, b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying

$$G(Tx, Ty, Tz) \leq k\max[G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)] \qquad (2.7)$$

for all $x, y, z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence x_n in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$G(x_n, x_{n+1}, x_{n+1}) = G(Tx_{n-1}, Tx_n, Tx_n)$$
$$\leq k\max[G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})]$$
$$\leq kG(x_{n-1}, x_n, x_n)$$
$$\leq k^n G(x_0, x_1, x_1).$$

Moreover for all $n, m \in N, n < m$ and by property (iv) in definition 1.3, we have

$$G(x_n, x_m, x_m) \leq s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)]$$
$$\leq s[k^n G(x_0, x_1, x_1) + G(x_{n+1}, x_m, x_m)]$$

$\leq sk^n G(x_0,x_1,x_1) + s^2[G(x_{n+1},x_{n+2},x_{n+2}) + G(x_{n+2},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2k^{n+1}G(x_0,x_1,x_1) + s^3[G(x_{n+2},x_{n+3},x_{n+3}) + G(x_{n+3},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2k^{n+1}G(x_0,x_1,x_1) + s^3k^{n+2}G(x_0,x_1,x_1) + s^3G(x_{n+3},x_m,x_m)]$

$\leq sk^n G(x_0,x_1,x_1) + s^2k^{n+1}G(x_0,x_1,x_1) + s^3k^{n+2}G(x_0,x_1,x_1)+\ldots+s^{m-1}k^{n+m-2}G(x_0,x_1,x_1) + s^{m-1}k^{n+m-1}G(x_0,x_1,x_1)$

$\leq sk^n[(1 + sk + (sk)^2 + (sk)^3+\ldots+(sk)^{m-2}) + (sk)^{m-2}k]G(x_0,x_1,x_1)$

$\leq sk^n[\frac{1-(sk)^{n-(m-2)}}{(1-sk)} + (sk)^{m-2}k]G(x_0,x_1,x_1).$

Letting $m,n \to \infty$, we have $\lim_{n,m\to\infty} G(x_n,x_m,x_m) = 0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that (x_n) is b_G-converges to u. We claim that u is fixed point of T. Consider

$G(x_n,Tu,Tu) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},Tu,Tu)]$

$\leq s[k^nG(x_0,x_1,x_1) + k\max[G(x_n,x_{n+1},x_{n+1}), G(u,Tu,Tu), G(u,Tu,Tu)]$

$\leq sk^nG(x_0,x_1,x_1) + skG(u,Tu,Tu).$

As $n \to \infty, x_n \to u$

$$G(u,Tu,Tu) \leq skG(u,Tu,Tu).$$

Since $s \geq 1$, we have $G(u,Tu,Tu) = 0$ and hence $Tu = u$ i.e. u is a fixed point of T.

Suppose $v \neq u$ is another fixed point of T i.e. $T(v) = v$. Consider

$G(x_n,Tv,Tv) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},Tv,Tv)]$

$G(x_n,v,v) \leq s[k^nG(x_0,x_1,x_1) + k\max[G(x_n,x_{n+1},x_{n+1}), G(v,v,v), G(v,v,v)]$

$\leq sk^nG(x_0,x_1,x_1) + skG(v,v,v).$

As $n \to \infty, x_n \to u$,

$$G(u,v,v) = 0,$$

since $s \geq 1$. This shows that $u = v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = u$. Consider

$G(u,Ty_n,Ty_n) \leq k\max[G(u,u,u), G(y_n,Ty_n,Ty_n), G(y_n,Ty_n,Ty_n)]$

$\leq kG(y_n,Ty_n,Ty_n).$

But

$G(y_n,Ty_n,Ty_n) \leq s[G(y_n,u,u) + G(u,Ty_n,Ty_n)]$

$\leq s[G(y_n,u,u) + kG(y_n,Ty_n,Ty_n)].$

As $n \to \infty$,

$G(u,Ty_n,Ty_n) \leq s[G(u,u,u) + kG(u,Ty_n,Ty_n)]$

$\leq skG(u,Ty_n,Ty_n).$

Since $s \geq 1$ and$k \in [0,1)$, we have $G(u,Ty_n,Ty_n) = 0$. Hence $Ty_n = u = T(u)$. This shows that T is b_G-continuous at u.

Corollary 2.8 Let (X,b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying for some $m \in N$

$$G(T^mx,T^my,T^mz) \leq k\max[G(x,T^mx,T^mx), G(y,T^my,T^my), G(z,T^mz,T^mz)] \quad (2.8)$$

for all $x,y,z \in X$ and $k \in [0,1)$. Then T has a unique fixed point (say u i.e. $Tu = u$) and T^m is b_G-continuous at u.

Theorem 2.9 Let (X,b_G) be a complete b_G-metric space with $s \geq 1$ and let $T: X \to X$ be a mapping satisfying

$$G(Tx,Ty,Tz) \leq a[G(x,Ty,Ty) + G(y,Tx,Tx)] \quad (2.9)$$

for all $x,y,z \in X$ and $as \in [0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu = u$)and T is b_G-continuous at u.

Proof: Let $x_0 \in X$. Define a sequence x_n in X such that $x_n = Tx_{n-1} = T^nx_0$. Consider

$G(x_n,x_{n+1},x_{n+1}) = G(Tx_{n-1},Tx_n,Tx_n)$

$\leq a[G(x_{n-1},x_{n+1},x_{n+1}) + G(x_n,x_n,x_n)]$

$\leq aG(x_{n-1},x_{n+1},x_{n+1})$

$\leq as[G(x_{n-1},x_n,x_n) + G(x_n,x_{n+1},x_{n+1})]$

$\leq \frac{as}{1-as}G(x_{n-1},x_n,x_n)$

$\leq kG(x_{n-1},x_n,x_n); k \in [0,1)$

$\leq k^nG(x_0,x_1,x_1).$

Moreover for all $n,m \in N, n < m$ and by property (iv) of definition 1.3, we have

$G(x_n,x_m,x_m) \leq s[G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_m,x_m)]$

$\leq s[k^nG(x_0,x_1,x_1)+G(x_{n+1},x_m,x_m)]$

$\leq sk^nG(x_0,x_1,x_1)+s^2[G(x_{n+1},x_{n+2},x_{n+2})+G(x_{n+2},x_m,x_m)]$

$\leq sk^nG(x_0,x_1,x_1)+s^2k^{n+1}G(x_0,x_1,x_1)+s^3[G(x_{n+2},x_{n+3},x_{n+3})+G(x_{n+3},x_m,x_m)]$

$\leq sk^nG(x_0,x_1,x_1)+s^2k^{n+1}G(x_0,x_1,x_1)+s^3k^{n+2}G(x_0,x_1,x_1)+s^3G(x_{n+3},x_m,x_m)]$

$\leq sk^nG(x_0,x_1,x_1)+s^2k^{n+1}G(x_0,x_1,x_1)+s^3k^{n+2}G(x_0,x_1,x_1)+\dots+s^{m-1}k^{n+m-2}G(x_0,x_1,x_1)+s^{m-1}k^{n+m-1}G(x_0,x_1,x_1)$

$\leq sk^n[(1+sk+(sk)^2+(sk)^3+\dots+(sk)^{m-2})+(sk)^{m-2}k]G(x_0,x_1,x_1)$

$\leq sk^n[\frac{1-(sk)^{n-(m-2)}}{(1-sk)}+(sk)^{m-2}k]G(x_0,x_1,x_1).$

Letting $m,n\to\infty$, we have $\lim_{n,m\to\infty}G(x_n,x_m,x_m)=0$. Hence (x_n) is a b_G-Cauchy sequence in X. Since X is complete, there exists $u\in X$ such that (x_n) is b_G converges to u. Now we claim that u is fixed point of T. Consider

$G(x_n,Tu,Tu)\leq s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},Tu,Tu)]$

$\leq s[k^nG(x_0,x_1,x_1)+a[G(x_n,Tu,Tu)+G(u,x_{n+1},x_{n+1})]]$

$\leq sk^nG(x_0,x_1,x_1)+asG(u,Tu,Tu)+asG(u,x_{n+1},x_{n+1})$.

As $n\to\infty$, we have $x_n\to u$ and

$$G(u,Tu,Tu)\leq skG(u,Tu,Tu).$$

Since $s\geq 1$ and $k\in[0,1)$, we have $G(u,Tu,Tu)=0$. Hence $Tu=u$. Therefore u is a fixed point of T.

Suppose $v\neq u$ is another fixed point of T i.e. $Tv=v$. Now

$G(x_n,Tv,Tv)\leq s[G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},Tv,Tv)]$

$G(x_n,v,v)\leq s[k^nG(x_0,x_1,x_1)+a[G(x_n,Tv,Tv)+G(v,x_n,x_n)]]$

$\leq sk^nG(x_0,x_1,x_1)+asG(x_n,v,v)+asG(v,x_n,x_n)$.

As $n\to\infty$, we have $x_n\to u$ and

$$G(u,v,v)\leq asG(u,v,v).$$

Since $s\geq 1$ and $k\in[0,1)$, we must have $G(u,v,v)=0$. Hence $u=v$.

Now we show that T is b_G-continuous at u. Let (y_n) be a sequence in X such that $\lim_{n\to\infty}y_n=u$. Consider

$$G(u,Ty_n,Ty_n)\leq a[G(u,Ty_n,Ty_n)+G(y_n,u,u)].$$

But

$G(y_n,Ty_n,Ty_n)\leq s[G(y_n,u,u)+a[G(u,Ty_n,Ty_n)+G(y_n,u,u)]]$

$\leq(1-a)sG(y_n,u,u)+asG(y_n,Ty_n,Ty_n)$.

As $n\to\infty$, we have

$G(u,Ty_n,Ty_n\leq(1-a)sG(u,u,u)+asG(u,Ty_n,Ty_n)$

$\leq asG(u,Ty_n,Ty_n)$.

Since $as<1/2$, we must have $G(u,Ty_n,Ty_n)=0$. Hence $Ty_n=Tu=u$. It shows that T is b_G-continuous at u.

Corollary 2.10 Let (X,b_G) be a complete b_G-metric space with $s\geq 1$ and let $T:X\to X$ be a mapping satisfying for some $m\in N$

$G(T^mx,T^my,T^mz)\leq a[G(x,T^my,T^my)+G(y,T^mx,T^mx)]$ (2.10)

for all $x,y,z\in X$ and where $as\in[0,1/2)$. Then T has a unique fixed point (say u i.e. $Tu=u$) and T^m is b_G-continuous at u.

Example 2.11 Let $X=$ R and define $G:R\times R\times R\to R$ *by* $G(x,y,z)=|x-y|+|y-z|+|x-z|$. Then (R,G) be a complete b_G-metric space. Let $T:R\to R$ defined by $T(x)=\frac{x}{3}$. Without loss of generality, we assume $x>y>z$. Then

(i)

$G(Tx,Ty,Tz)=|\frac{x}{3}-\frac{y}{3}|+|\frac{y}{3}-\frac{z}{3}|+|\frac{x}{3}-\frac{z}{3}|$

$=\frac{1}{3}[|x-y|+|y-z|+|x-z|]$

$\leq kG(x,y,z), k\in[0,1)$.

(ii)

$G(Tx,Ty,Tz)=|\frac{x}{3}-\frac{y}{3}|+|\frac{y}{3}-\frac{z}{3}|+|\frac{x}{3}-\frac{z}{3}|$

$\leq\frac{x}{3}+\frac{y}{3}+\frac{x}{3}$

$\leq|x-\frac{x}{3}|+|y-\frac{y}{3}|$

$\leq\frac{1}{2}[2|x-\frac{x}{3}|+2|y-\frac{y}{3}|]$

$\leq k[G(x,T(x),T(x))+G(y,T(y),T(y))]$.

(iii)

$G(Tx,Ty,Tz)=|\frac{x}{3}-\frac{y}{3}|+|\frac{y}{3}-\frac{z}{3}|+|\frac{x}{3}-\frac{z}{3}|$

$\leq\frac{x}{3}+\frac{x}{3}$

$\leq\frac{1}{2}[2|x-\frac{x}{3}|]$

$\leq k\max\{G(x,y,z),G(x,T(x),T(x)),G(y,T(y),T(y))\}$.

(iv)

$$G(Tx,Tx,Tx) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}|$$
$$\leq \frac{x}{3} + \frac{x}{3}$$
$$\leq \frac{1}{2}[2|x - \frac{x}{3}|]$$
$$\leq$$
$$k\max\{G(x,T(x),T(x)), G(y,T(y),T(y)), G(z,T(z),T(z))\}.$$

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FIXED POINT THEOREMS SATISFY PROPERTY P IN G_b-METRIC SPACES

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Abstract. In this paper, we prove fixed point theorem for a contractive mapping which satisfy property P in G_b-metric space. Our results are supported by an example.

1. Introduction

The metric space is generalized by different authors by various ways. Czerwik [9] introduced b-metric space. Zead Mustafa and Brailey Sims [13] coined the concept of G-metric space. A. Aghajani, M. Abbas and J. R. Roshan [2] extended the G-metric space with b-metric space and develop the new structure of metric space, which we call G_b-metric space. They proved the fixed point theorems in G_b-metric spaces. A self map T of the space X with a nonempty fixed point set $F(T)$. Then we say that T has a property P if $F(T) = F(T^n)$ for each $n \in N$ [10]-[3]. We design the fixed point theorems for the self maps which satisfy property P for various contractions in G-metric spaces [10]- [8].

Definition 1 ([9]) Let X be a non empty set and the mapping $d : X \times X \to [0, \infty)$. The mapping d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. Then d is called a b-metric on X. The ordered pair (X, d) is called b-metric space with coefficient s.

G-metric space is defined as follows

Definition 2 ([13])Let X be a non empty set and the mapping $G : X \times X \times X \to [0, \infty)$. The mapping G satisfies

(i) $G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$;
(ii) $0 < G(x, x, y)$ for all $x, y \in X$;
(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

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(v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality). Then G is called a G-metric on X and the ordered pair (X,G) is called G-metric space.

A. Aghajani, M. Abbas and J. R. Roshan [2] introduced G_b- metric space as follows

Definition 3 ([2]) Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G_b : X \times X \times X \to R^+$ satisfies:

(i) $G_b(x,y,z) = 0$ if $x = y = z$ for all $x,y,z \in X$;
(ii) $0 < G_b(x,x,y)$ for all $x,y,z \in X$ with $x \neq y$;
(iii) $G_b(x,x,y) \leq G_b(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$;
(iv) $G_b(x,y,z) = G_b(px,z,y)$, where p is a permutation of x,y,z (symmetry);
(v) $G_b(x,y,z) \leq s[G_b(x,a,a) + G_b(a,y,z)]$.

Then G_b is called a generalized b -metric or G_b metric on X. The ordered pair (X,G_b) is called generalized b metric or G_b-metric space.

Following example shows that a G_b-metric on X need not be a G-metric on X.

Example 1 ([2]) Let (X,G) be a G-metric space and $G_*(x,y,z) = G(x,y,z)^p$; where $p > 1$ is a real number. Note that G_* is a G_b-metric with $s = 2^{P-1}$. Obviously, G_* satisfies conditions (i) to (iv) of the G_b metric space. Now it suffices to show that condition (v) of G_b metric space to be hold. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$. Thus for each $x,y,z,a \in X$ we obtain

$$\begin{aligned} G_*(x,y,z) = G(x,y,z)^p &\leq (G(x,a,a) + G(a,y,z))^p \\ &\leq 2^{p-1}(G(x,a,a)^p + G(a,y,z)^p) \\ &= 2^{p-1}(G_*(x,a,a) + G_*(a,y,z)). \end{aligned}$$

So G_* is a G_b-metric with $s = 2^{p-1}$.

Also in the above example, (X,G_*) is not necessarily a G-metric space.

Example 2 Let $X = R$ and let

$$G_b(x,y,z) = \max\{|x-y|^2, |y-z|^2, |x-z|^2\}.$$

Then (X,G_b) is a G_b-metric space with the coefficient $s = 2$.

Definition 4 ([2]) Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be :

(i) G_b-Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m,n,l \geq n_0$, $G(x_n,x_m,x_l) < \epsilon$;
(ii) G_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m,n \geq n_0$, $G(x_n,x_m,x) < \epsilon$.

Proposition 1 ([2]) Let (X,G_b) be a G_b-metric space. Then the following are equivalent:

(i) $\{x_n\}$ is G_b-converges to x.
(ii) $G_b(x_n,x_n,x) \to 0$, as $n \to \infty$.
(iii) $G_b(x_n,x,x) \to 0$, as $n \to \infty$.

Proposition 2 ([2]) Let (X,G_b) be a G_b-metric space. Then the following are equivalent:

(i) The sequence $\{x_n\}$ is G_b-Cauchy.
(ii) For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n,x_m,x_l) < \epsilon$, for all $n,m,l \geq n_0$.

Definition 5 ([2]) A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

2. Main results

Our first main result is

Theorem 1 Let (X, G_b) be a complete G_b-metric space with $s \geq 1$ and let $T : X \to X$ be a mapping satisfying

$$\begin{aligned} G_b(Tx,Ty,Tz) \leq k \max \Big[& G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz), \\ & \frac{[G_b(x,Ty,Ty)+G_b(y,Tx,Tx)]}{2}, \frac{[G_b(y,Tz,Tz)+G_b(z,Ty,Ty)]}{2}, \\ & \frac{[G_b(x,Tz,Tz)+G_b(z,Tx,Tx)]}{2} \Big]; \end{aligned} \tag{1}$$

for all $x, y, z \in X$, where k is such that $sk \in [0, 1)$. Then T has a unique fixed point say p $(Tp = p)$ in X and T is G_b-continuous at p.

Proof. Let $x_0 \in X$. Since $T : X \to X$ be a self map, then we get a sequence (x_n) in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$\begin{aligned} G_b(x_n,x_{n+1},x_{n+1}) &= G_b(Tx_{n-1},Tx_n,Tx_n) \\ &\leq k \max \Big[G_b(x_{n-1},x_n,x_n), G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), \\ &\quad G_b(x_n,x_{n+1},x_{n+1}), \frac{[G_b(x_{n-1},x_{n+1},x_{n+1})+G_b(x_n,x_n,x_n)]}{2}, \\ &\quad \frac{[G_b(x_n,x_{n+1},x_{n+1})+G_b(x_n,x_{n+1},x_{n+1})]}{2}, \\ &\quad \frac{[G_b(x_{n-1},x_{n+1},x_{n+1})+G_b(x_n,x_n,x_n)]}{2} \Big] \\ &\leq k \max \Big[G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), \\ &\quad \frac{G_b(x_{n-1},x_{n+1},x_{n+1})}{2} \Big]. \end{aligned}$$

There are three cases:

Case (i) Suppose

$$\max \Big[G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), \frac{G_b(x_{n-1},x_{n+1},x_{n+1})}{2} \Big] = \frac{G_b(x_{n-1},x_{n+1},x_{n+1})}{2}.$$

Then

$$G_b(x_n,x_{n+1},x_{n+1}) \leq k \frac{G_b(x_{n-1},x_{n+1},x_{n+1})}{2}.$$

By property (v) of G_b metric space, we have

$$G_b(x_{n-1},x_{n+1},x_{n+1}) \leq s\{G_b(x_{n-1},x_n,x_n)+G_b(x_n,x_{n+1},x_{n+1})\}.$$

Then, we get

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\leq sk\Big[\frac{G_b(x_{n-1}, x_n, x_n) + G_b(x_n, x_{n+1}, x_{n+1})}{2}\Big] \\ &\leq \frac{sk}{(2-sk)} G_b(x_{n-1}, x_n, x_n) \\ &= \lambda G_b(x_{n-1}, x_n, x_n), \end{aligned}$$

where $\lambda = \frac{sk}{(2-sk)}$. Therefore by continuing in this way, we get

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_b(x_0, x_1, x_1).$$

As $n \to \infty, G_b(x_n, x_{n+1}, x_{n+1}) \to 0$, since $\lambda < 1$. Moreover for all $n, m \in \mathbb{N}, n < m$, and by (v) the property of G_b-metric space.

$$\begin{aligned} G_b(x_n, x_m, x_m) &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m)] \\ &\leq s[\lambda^n G_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m)] \\ &\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2[G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m)] \\ &\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3[G_b(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + G_b(x_{n+3}, x_m, x_m)] \\ &= s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3\lambda^{n+2} G_b(x_0, x_1, x_1) + \\ &\quad s^3 G_b(x_{n+3}, x_m, x_m) \\ &\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3\lambda^{n+2} G_b(x_0, x_1, x_1) \\ &\quad + \cdots + s^{m-1}\lambda^{n+m-2} G_b(x_0, x_1, x_1) + s^{m-1}\lambda^{n+m-1} G_b(x_0, x_1, x_1) \\ &= s\lambda^n[(1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + ... + (s\lambda)^{m-2}) + (s\lambda)^{m-2}\lambda] \\ &\quad G_b(x_0, x_1, x_1) \\ &= s\lambda^n[\frac{1-(s\lambda)^{n-(m-2)}}{(1-s\lambda)} + (s\lambda)^{m-2}\lambda] G_b(x_0, x_1, x_1). \end{aligned}$$

Letting $m, n \to \infty$, we get

$$\lim_{n,m\to\infty} G_b(x_n, x_m, x_m) = 0.$$

Case (ii) Suppose

$$\max\Big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}\Big] = G_b(x_n, x_{n+1}, x_{n+1}).$$

Then

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq k G_b(x_n, x_{n+1}, x_{n+1}),$$

which is contradiction, since $k < 1$.
Case (iii) Suppose

$$\max\Big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), \frac{G_b(x_{n-1}, x_{n+1}, x_{n+1})}{2}\Big] = G_b(x_{n-1}, x_n, x_n).$$

Then

$$G_b(x_n, x_{n+1}, x_{n+1}) \leq kG_b(x_{n-1}, x_n, x_n)$$
$$\leq k^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \leq \cdots \leq k^n G_b(x_0, x_1, x_1).$$

As $n \to \infty, G_b(x_n, x_{n+1}, x_{n+1}) \to 0$, since $k < 1$. Also Since $sk < 1$ and by case(i) $\{x_n\}$ is a G_b -Cauchy sequence in X. Since X is G_b-complete, then there exists $p \in X$ such that $\{x_n\}$ is G_b-converges to $p \in X$. Now we claim that p is fixed point of T. Suppose that $Tp \neq p$.

$$G_b(x_n, Tp, Tp) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)]$$
$$\leq s\lambda^n G_b(x_0, x_1, x_1) + sk \max \Big[G_b(x_n, p, p), G_b(x_n, x_{n+1}, x_{n+1}),$$
$$G_b(p, Tp, Tp), G_b(p, Tp, Tp), \frac{[G_b(x_n, Tp, Tp) + G_b(Tp, x_{n+1}, x_{n+1})]}{2},$$
$$\frac{[G_b(p, Tp, Tp) + G_b(p, Tp, Tp)]}{2},$$
$$\frac{[G_b(x_n, Tp, Tp) + G_b(Tp, x_{n+1}, x_{n+1})]}{2}\Big].$$

As $n \to \infty, \{x_n\} \to p$ and above inequality terns into

$$G_b(p, Tp, Tp) \leq skG_b(p, Tp, Tp).$$

It is contradiction, since $sk < 1$. Thus $Tp = p$. Therefore p is a fixed point of T. For uniqueness suppose $q \neq p$ and q is another fixed point of T, i.e. $Tq = q$. By (v) the property of G_b-metric space

$$G_b(x_n, Tq, Tq) \leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)].$$

Therefore

$$G_b(x_n, q, q) \leq s\lambda^n G_b(x_0, x_1, x_1) + sk \max \Big[G_b(x_n, q, q), G_b(x_n, x_{n+1}, x_{n+1}),$$
$$G_b(q, Tq, Tq), G_b(q, Tq, Tq), \frac{[G_b(x_n, Tq, Tq) + G_b(q, x_{n+1}, x_{n+1})]}{2},$$
$$\frac{[G_b(q, Tq, Tq) + G_b(q, Tq, Tq)]}{2}, \frac{[G_b(x_n, Tq, Tq) + G_b(q, x_{n+1}, x_{n+1})]}{2}\Big].$$

As $\lambda < 1$, we extend $n \to \infty$, so $\{x_n\} \to p$. Thus we get

$$G_b(p, q, q) \leq sk \max \Big[\frac{[G_b(p, q, q) + G_b(q, p, p)]}{2}, G_b(p, q, q)\Big].$$

There are two cases

Case (i) Suppose

$$\max \Big[\frac{[G_b(p, q, q) + G_b(q, p, p)]}{2}, G_b(p, q, q)\Big] = G_b(p, q, q).$$

Then

$$G_b(p, q, q) \leq skG_b(p, q, q),$$

which is contradiction, since $sk < 1$.

Case (ii) Suppose

$$\max\Big[\frac{[G_b(p,q,q)+G_b(q,p,p)]}{2}, G_b(p,q,q)\Big] = \frac{[G_b(p,q,q)+G_b(q,p,p)]}{2}.$$

Then

$$G_b(p,q,q) \le sk\Big[\frac{G_b(p,q,q)+G_b(q,p,p)}{2}\Big].$$

It implies that

$$G_b(p,q,q) \le \frac{sk}{2-sk} G_b(q,p,p). \tag{2}$$

Also consider,

$$\begin{aligned} G_b(Tq,x_n,x_n) \le k\max\Big[&G_b(q,x_{n-1},x_{n-1}), G_b(q,Tq,Tq), G_b(x_{n-1},x_n,x_n),\\ &G_b(x_{n-1},x_n,x_n), \frac{[G_b(q,x_n,x_n)+G_b(x_{n-1},Tq,Tq)]}{2},\\ &\frac{[G_b(x_{n-1},x_n,x_n)+G_b(x_{n-1},x_n,x_n)]}{2},\\ &\frac{[G_b(q,x_n,x_n)+G_b(x_n,q,q)]}{2}\Big]. \end{aligned}$$

As $n\to\infty$, we get

$$G_b(q,p,p) \le k\max\Big[G_b(q,p,p), \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}\Big].$$

There are two cases:
Case (i)Suppose

$$\max\Big[G_b(q,p,p), \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}\Big] = G_b(q,p,p).$$

Then, we get

$$G_b(q,p,p) \le kG_b(q,p,p),$$

which is contradiction, since $k<1$.

Case (ii) Suppose

$$\max\Big[G_b(q,p,p), \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}\Big] = \frac{[G_b(q,p,p)+G_b(p,q,q)]}{2}.$$

$$G_b(q,p,p) \le k\Big[G_b(q,p,p) + \frac{G_b(p,q,q)}{2}\Big].$$

It implies that

$$G_b(q,p,p) \le \Big(\frac{k}{2-k}\Big) G_b(p,q,q). \tag{3}$$

Therefore from (2) and (3), we get

$$G_b(p,q,q) \le \Big(\frac{sk}{2-sk}\Big)\Big(\frac{k}{2-k}\Big) G_b(p,q,q). \tag{4}$$

Since $(\frac{sk}{2-sk})(\frac{k}{2-k}) < 1$. So (2.4) is possible only when $G_b(p,q,q) = 0$. Thus $p = q$. Now we claim that T is G_b-continuous at p. Let (y_n) be a sequence in X such that $\lim_{n\to\infty} y_n = p$. Consider

$$\begin{aligned} G_b(p,Ty_n,Ty_n) \le k\max\Big[&G_b(p,y_n,y_n), G_b(p,p,p), G_b(y_n,Ty_n,Ty_n),\\ &G_b(y_n,Ty_n,Ty_n), \frac{[G_b(p,Ty_n,Ty_n)+G_b(y_n,p,p)]}{2},\\ &\frac{[G_b(y_n,Ty_n,Ty_n)+G_b(y_n,Ty_n,Ty_n)]}{2},\\ &\frac{[G_b(p,Ty_n,Ty_n)+G_b(y_n,p,p)]}{2}\Big]. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} G_b(p,Ty_n,Ty_n) &\le k\max\Big[G_b(p,Ty_n,Ty_n), \frac{G_b(p,Ty_n,Ty_n)}{2}\Big]\\ &= kG_b(p,Ty_n,Ty_n). \end{aligned}$$

Since $k < 1$, It is possible only when $G_b(p,Ty_n,Ty_n) = 0$ i.e. $Ty_n = p = Tp$. Therefore T is G_b-continuous at p.

Theorem 2 Let (X,G_b) be a complete G_b-metric space with $s \ge 1$ and let $T : X \to X$ be a mapping satisfying

$$\begin{aligned} G_b(Tx,Ty,Tz) \le k\max\Big[&G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(x,Ty,Ty),\\ &G_b(y,Tx,Tx), G_b(z,Tz,Tz),\Big]; \end{aligned} \tag{5}$$

for all $x,y,z \in X$ and where k is such that $sk \in [0,\frac{1}{2})$. Then T has a unique fixed point say p in X(i.e. $Tp = p$)and T is G_b-continuous at p.

Proof. Let $x_0 \in X$ and $T : X \to X$ be a self map. Then, we get a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$. Consider

$$\begin{aligned} G_b(x_n,x_{n+1},x_{n+1}) &= G_b(Tx_{n-1},Tx_n,Tx_n)\\ &\le k\max\Big[G_b(x_{n-1},x_n,x_n), G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}),\\ &\quad G_b(x_{n-1},x_{n+1},x_{n+1}), G_b(x_n,x_n,x_n), G_b(x_n,x_{n+1},x_{n+1})\Big]\\ &= k\max\Big[G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}),\\ &\quad G_b(x_{n-1},x_{n+1},x_{n+1})\Big]. \end{aligned}$$

There are three cases:

Case (i) Suppose

$$\begin{aligned} \max\Big[G_b(x_{n-1},x_n,x_n), G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n-1},x_{n+1},x_{n+1})\Big]\\ = G_b(x_{n-1},x_{n+1},x_{n+1}). \end{aligned}$$

Then, we get

$$G_b(x_n,x_{n+1},x_{n+1}) \le kG_b(x_{n-1},x_{n+1},x_{n+1}).$$

Then by property (v) of G_b metric space, we have

$$G_b(x_{n-1},x_{n+1},x_{n+1}) \le s\{G_b(x_{n-1},x_n,x_n) + G_b(x_n,x_{n+1},x_{n+1})\}.$$

Thus

$$G_b(x_n,x_{n+1},x_{n+1}) \le sk\Big[G_b(x_{n-1},x_n,x_n)+G_b(x_n,x_{n+1},x_{n+1})\Big].$$

It gives that

$$\begin{aligned} G_b(x_n,x_{n+1},x_{n+1}) &\le \frac{sk}{(1-sk)}G_b(x_{n-1},x_n,x_n) \\ &= \lambda G_b(x_{n-1},x_n,x_n). \end{aligned}$$

where $\lambda = \dfrac{sk}{1-sk}$. Therefore by continuing in this way, we get

$$G_b(x_n,x_{n+1},x_{n+1}) \le \lambda^n G_b(x_0,x_1,x_1).$$

Since $k<1$, letting $n\to\infty$, we have $n\to\infty, G_b(x_n,x_{n+1},x_{n+1})\to 0$. Moreover for all $n,m\in\mathbb{N}, n<m$, since $k<\lambda<1$ and by (v) the property of G_b metric space, we have

$$\begin{aligned} G_b(x_n,x_m,x_m) &\le s\Big[G_b(x_n,x_{n+1},x_{n+1})+G_b(x_{n+1},x_m,x_m)\Big] \\ &\le s\Big[\lambda^n G_b(x_0,x_1,x_1)+G_b(x_{n+1},x_m,x_m)\Big] \\ &\le s\lambda^n G_b(x_0,x_1,x_1)+s^2\Big[G_b(x_{n+1},x_{n+2},x_{n+2})+G_b(x_{n+2},x_m,x_m)\Big] \\ &\le s\lambda^n G_b(x_0,x_1,x_1)+s^2\lambda^{n+1}G_b(x_0,x_1,x_1) \\ &\quad + s^3\Big[G_b(x_{n+2},x_{n+3},x_{n+3})+G_b(x_{n+3},x_m,x_m)\Big] \\ &\le s\lambda^n G_b(x_0,x_1,x_1)+s^2\lambda^{n+1}G_b(x_0,x_1,x_1)+s^3\lambda^{n+2}G_b(x_0,x_1,x_1) \\ &\quad +\cdots+s^{m-1}\lambda^{n+m-2}G_b(x_0,x_1,x_1)+s^{m-1}\lambda^{n+m-1}G_b(x_0,x_1,x_1) \\ &= s\lambda^n\Big[(1+s\lambda+(s\lambda)^2+(s\lambda)^3+\cdots+(s\lambda)^{m-2})+(s\lambda)^{m-2}\lambda\Big] \\ &\quad G_b(x_0,x_1,x_1) \\ &= s\lambda^n\Big[\frac{1-(s\lambda)^{n-(m-2)}}{(1-s\lambda)}+(s\lambda)^{m-2}\lambda\Big]G_b(x_0,x_1,x_1). \end{aligned}$$

Letting $m,n\to\infty$, we get, $\lim_{n,m\to\infty} G_b(x_n,x_m,x_m)=0$, since $sk<1$. This show that $\{x_n\}$ is a G_b-Cauchy sequence in X.
Case (ii) Suppose

$$\begin{aligned} &\max\Big[G_b(x_{n-1},x_n,x_n),G_b(x_n,x_{n+1},x_{n+1}),G_b(x_{n-1},x_{n+1},x_{n+1})\Big] \\ &\qquad = G_b(x_n,x_{n+1},x_{n+1}). \end{aligned}$$

Then

$$G_b(x_n,x_{n+1},x_{n+1}) \le kG_b(x_n,x_{n+1},x_{n+1}),$$

which is a contradiction, since $k<1$.
Case (iii) Suppose

$$\begin{aligned} &\max\Big[G_b(x_{n-1},x_n,x_n),G_b(x_n,x_{n+1},x_{n+1}),G_b(x_{n-1},x_{n+1},x_{n+1})\Big] \\ &\qquad = G_b(x_{n-1},x_n,x_n). \end{aligned}$$

Then

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &\le kG_b(x_{n-1}, x_n, x_n) \\ &\le k^2 G_b(x_{n-2}, x_{n-1}, x_{n-1}) \le \cdots \le k^n G_b(x_0, x_1, x_1). \end{aligned}$$

Since $k < 1$, as $n \to \infty$, we have $G_b(x_n, x_{n+1}, x_{n+1}) \to 0$. Thus in this case also $\{x_n\}$ is a G_b-Cauchy sequence in X. Since X is G_b-complete, then there exists $p \in X$ such that $\{x_n\} \to p$. Now, we claim that p is fixed point of T. Suppose that $Tp \neq p$, by (v) the property of G_b-metric space and by (2.5), we get

$$\begin{aligned} G_b(x_n, Tp, Tp) &\le s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)] \\ &\le s\lambda^n G_b(x_0, x_1, x_1) + sk \max\Big[G_b(x_n, p, p), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(p, Tp, Tp), G_b(x_n, Tp, Tp), G_b(p, x_{n+1}, x_{n+1}), G_b(p, Tp, Tp)\Big]. \end{aligned}$$

Letting $n \to \infty$, we have $\{x_n\} \to p$. Then

$$G_b(p, Tp, Tp) \le skG_b(p, Tp, Tp).$$

Since $sk < 1$, the above inequality is true only if $G_b(p, Tp, Tp) = 0$. Thus $p = Tp$. Therefore p is a fixed point of T. For uniqueness, suppose $q \neq p$ and q is another fixed point of T i.e. $Tq = q$. Consider

$$G_b(x_n, Tq, Tq) \le s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)].$$

It gives that

$$\begin{aligned} G_b(x_n, q, q) &\le s\lambda^n G_b(x_0, x_1, x_1) + sk \max\Big[G_b(x_n, q, q), G_b(x_n, x_{n+1}, x_{n+1}), \\ &\quad G_b(q, Tq, Tq), G_b(x_n, Tq, Tq), G_b(q, x_{n+1}, x_{n+1}), G_b(q, Tq, Tq)\Big]. \end{aligned}$$

Letting $n \to \infty$, we have $\{x_n\} \to p$ with $Tq = q$. Then, we get

$$G_b(p, q, q) \le sk \max\Big[G_b(p, q, q), G_b(q, p, p)\Big].$$

There are two cases:

Case (a) Suppose $\max\Big[G_b(p, q, q), G_b(q, p, p)\Big] = G_b(p, q, q)$. Then

$$G_b(p, q, q) \le skG_b(p, q, q),$$

which is contradiction, since $sk < 1$.

Case (b) Suppose $\max\Big[G_b(p, q, q), G_b(q, p, p)\Big] = G_b(q, p, p)$. Then

$$G_b(p, q, q) \le skG_b(q, p, p). \tag{6}$$

Now, consider

$$\begin{aligned} G_b(Tq, x_n, x_n) &\le k \max\Big[G_b(q, x_{n-1}, x_{n-1}), G_b(q, Tq, Tq), G_b(x_{n-1}, x_n, x_n), \\ &\quad G_b(q, x_n, x_n), G_b(x_{n-1}, Tq, Tq), G_b(x_{n-1}, x_n, x_n)\Big]. \end{aligned}$$

Letting $n \to \infty$, it implies that

$$G_b(q, p, p) \le k \max\Big[G_b(q, p, p), G_b(p, q, q)\Big].$$

There are two cases:

Case (c) Suppose $\max\Big[G_b(q,p,p),G_b(p,q,q)\Big]=G_b(q,p,p)$. Then

$$G_b(q,p,p)\le kG_b(q,p,p),$$

which is contradiction, since $k<1$.

Case (d) Suppose $\max\Big[G_b(q,p,p),G_b(p,q,q)\Big]=G_b(p,q,q)$. Then

$$G_b(q,p,p)\le kG_b(p,q,q). \tag{7}$$

Using inequality (7) in (6), we have

$$G_b(p,q,q)\le sk^2G_b(p,q,q). \tag{8}$$

Since $sk<1$. Thus (8) is true only if $G_b(p,q,q)=0$. Thus $p=q$. Therefore p is a fixed point of T in X.

To show that T is G_b-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty}y_n=p$. Consider

$$\begin{aligned}G_b(p,Ty_n,Ty_n)\le k\max\Big[&G_b(p,y_n,y_n),G_b(p,p,p),G_b(y_n,Ty_n,Ty_n),\\&G_b(p,Ty_n,Ty_n),G_b(y_n,p,p),G_b(y_n,Ty_n,Ty_n)\Big].\end{aligned}$$

Letting $n\to\infty$, we get

$$\begin{aligned}G_b(p,Ty_n,Ty_n)\le\max\Big[&G_b(p,p,p),G_b(p,p,p),G_b(p,Ty_n,Ty_n),G_b(p,Ty_n,Ty_n),\\&G_b(p,p,p),G_b(p,Ty_n,Ty_n)\Big].\end{aligned}$$

Thus

$$G_b(p,Ty_n,Ty_n)\le k\max G_b(p,Ty_n,Ty_n).$$

It is possible only if $G_b(p,Ty_n,Ty_n)=0$. Thus $Ty_n=p=Tp$. It is proved that T is G_b-continuous at p.

Theorem 3 Let (X,G_b) be a complete G_b-metric space with $s\ge1$ and let $T:X\to X$ be a mapping satisfying

$$G_b(Tx,Ty,Tz)\le\alpha G_b(x,y,z)+\beta G_b(x,Tx,Tx)+\gamma G_b(y,Ty,Ty)+\delta G_b(z,Tz,Tz) \tag{9}$$

for all $x,y,z\in X$ and where $\alpha+\beta+\gamma+\delta<1$. Then T has a unique fixed point say p (i.e. $Tp=p$) and T is G_b-continuous at p.

Proof. Let $x_0\in X$ and the mapping $T:X\to X$ then we get a sequence $\{x_n\}$ in X such that $x_n=Tx_{n-1}=T^nx_0$. Consider

$$\begin{aligned}G_b(x_n,x_{n+1},x_{n+1})&=G_b(Tx_{n-1},Tx_n,Tx_n)\\&\le\alpha G_b(x_{n-1},x_n,x_n)+\beta G_b(x_{n-1},x_n,x_n)+\gamma G_b(x_n,x_{n+1},x_{n+1})\\&\quad+\delta G_b(x_n,x_{n+1},x_{n+1})\\&\le(\alpha+\beta)G_b(x_{n-1},x_n,x_n)+(\gamma+\delta)G_b(x_n,x_{n+1},x_{n+1})\\&\le\frac{\alpha+\beta}{1-(\gamma+\delta)}G_b(x_{n-1},x_n,x_n)\\&\le\lambda G_b(x_{n-1},x_n,x_n),\end{aligned}$$

where $\lambda=\frac{\alpha+\beta}{1-(\gamma+\delta)}$. Therefore, continuing in this way, we get $G_b(x_n,x_{n+1},x_{n+1})\le\lambda^nG_b(x_0,x_1,x_1)$.

Moreover for all $n, m \in \mathbb{N}, n < m$ and by (v) th property of G_b metric space, we have

$$\begin{aligned}
G_b(x_n, x_m, x_m) &\leq s\Big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_m, x_m)\Big] \\
&\leq s\Big[\lambda^n G_b(x_0, x_1, x_1) + G_b(x_{n+1}, x_m, x_m)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\Big[G_b(x_{n+1}, x_{n+2}, x_{n+2}) + G_b(x_{n+2}, x_m, x_m)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) \\
&\quad + s^3\Big[G_b(x_{n+2}, x_{n+3}, x_{n+3}) + G_b(x_{n+3}, x_m, x_m)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s^2\lambda^{n+1} G_b(x_0, x_1, x_1) + s^3\lambda^{n+2} G_b(x_0, x_1, x_1) + \\
&\quad \cdots + s^{m-1}\lambda^{n+m-2} G_b(x_0, x_1, x_1) + s^{m-1}\lambda^{n+m-1} G_b(x_0, x_1, x_1) \\
&= s\lambda^n\Big[(1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + \cdots + (s\lambda)^{m-2}) + (s\lambda)^{m-2}\lambda\Big] \\
&\quad G_b(x_0, x_1, x_1) \\
&= s\lambda^n\Big[\frac{1 - (s\lambda)^{n-(m-2)}}{(1 - s\lambda)} + (s\lambda)^{m-2}\lambda\Big] G_b(x_0, x_1, x_1).
\end{aligned}$$

Letting $m, n \to \infty$, we have $\lim_{n,m\to\infty} G_b(x_n, x_m, x_m) = 0$. Hence $\{x_n\}$ is a G_b -Cauchy sequence in X. Since X is a G_b-complete, therefore there exists $p \in X$ such that $\{x_n\}$ is G_b -converges to p. Now we will show here p is fixed point of T. Suppose that $Tp \neq p$.

$$\begin{aligned}
G_b(x_n, Tp, Tp) &\leq s\Big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)\Big] \\
&\leq s\lambda^n G_b(x_0, x_1, x_1) + s\Big[\alpha G_b(x_n, p, p) + \beta G_b(x_n, x_{n+1}, x_{n+1}) \\
&\quad + \gamma G_b(p, Tp, Tp) + \delta G_b(p, Tp, Tp)\Big].
\end{aligned}$$

Letting $n \to \infty$, since $\lambda < 1$, so $\lambda^n \to 0$ and $x_n \to p$. It gives that

$$G_b(p, Tp, Tp) \leq s(\gamma + \delta) G_b(p, Tp, Tp).$$

Since $s(\gamma + \delta) < 1$. The above inequality is true only if $G_b(p, Tp, Tp) = 0$ i.e. $p = Tp$. Thus p is a fixed point of T.

Suppose $q \neq p$ and q is another fixed point of T, i.e. $Tq = q$. Then consider

$$\begin{aligned}
G_b(x_n, Tq, Tq) &\leq s\Big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)\Big] \\
G_b(x_n, q, q) &\leq s\lambda^n G_b(x_0, x_1, x_1) + s\Big[\alpha G_b(x_n, q, q) + \beta G_b(x_n, x_{n+1}, x_{n+1}) + \\
&\quad \gamma G_b(q, Tq, Tq) + \delta G_b(q, Tq, Tq)\Big].
\end{aligned}$$

As $n \to \infty, x_n \to p$, $\lambda^n \to 0$ as $\lambda < 1$ and $Tq = q$. We get

$$G_b(p, q, q) \leq s\alpha G_b(p, q, q), \tag{10}$$

since $s\alpha < 1$. The inequality (10) is true only when $G_b(p, q, q) = 0$. i.e.$p = q$. Thus p is a unique fixed point of T.

To show that T is G_b-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = p$. Consider

$$\begin{aligned}G_b(p, Ty_n, Ty_n) &\leq \alpha G_b(p, y_n, y_n) + \beta G_b(p, p, p) + \gamma G_b(y_n, Ty_n, Ty_n)\\ &\quad + \delta G_b(y_n, Ty_n, Ty_n)\\ &= \alpha G_b(p, y_n, y_n) + (\gamma+\delta) G_b(y_n, Ty_n, Ty_n)\\ &\leq \alpha G_b(p, y_n, y_n) + s(\gamma+\delta)\{G_b(y_n, p, p) + G_b(p, Ty_n, Ty_n)\}.\end{aligned}$$

It is implies that

$$G_b(p, Ty_n, Ty_n) \leq \Big[\frac{\alpha}{1-s(\gamma+\delta)} G_b(p, y_n, y_n) + \frac{s(\gamma+\delta)}{1-s(\gamma+\delta)} G_b(p, Ty_n, Ty_n)\Big].$$

As $n \to \infty, \{y_n\} \to p$, we get

$$G_b(p, Ty_n, Ty_n) = \frac{s(\gamma+\delta)}{1-s(\gamma+\delta)} G_b(p, Ty_n, Ty_n). \tag{11}$$

Since $\frac{s(\gamma+\delta)}{1-s(\gamma+\delta)} < 1$. The inequality (11) is true only if $G_b(p, Ty_n, Ty_n) = 0$. i.e. $Ty_n = p = Tp$, as $n \to \infty$. It shows that T is G_b-continuous at p.

3. Property P

Let T be a self map of a complete G_b metric space with non-empty fixed point set $F(T)$. Then T is said to satisfy property P if $F(T) = F(T^n)$, for each $n \in \mathbb{N}$.

Theorem 4 Under the contraction of theorem 1, T has property P.

Proof. By Theorem 1, T has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $p \neq Tp$. Then we have

$$\begin{aligned}G_b(p, Tp, Tp) &= G_b(T^n p, T^{n+1}p, T^{n+1}p)\\ &\leq k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^{n-1}p, T^n p, T^n p),\\ &\quad G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^n p, T^{n+1}p, T^{n+1}p),\\ &\quad \frac{[G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) + G_b(T^n p, T^n p, T^n p)]}{2},\\ &\quad \frac{[G_b(T^n p, T^{n+1}p, T^{n+1}p) + G_b(T^n p, T^{n+1}p, T^{n+1}p)]}{2},\\ &\quad \frac{[G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) + G_b(T^n p, T^n p, T^n p)]}{2}\Big]\\ &= k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p),\\ &\quad \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big].\end{aligned}$$

Here, we have three cases:

Case (i) Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big] = G_b(T^{n-1}p, T^n p, T^n p).$$

Then, we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq kG_b(T^{n-1}p, T^np, T^np) \leq \cdots \leq k^nG_b(p, Tp, Tp).$$

Case (ii) Suppose

$$\max\Big[G_b(T^{n-1}p, T^np, T^np), G_b(T^np, T^{n+1}p, T^{n+1}p), \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big] = G_b(T^np, T^{n+1}p, T^{n+1}p).$$

Then, we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq kG_b(T^np, T^{n+1}p, T^{n+1}p),$$

which is contradiction, since $k < 1$.
Case (iii) Suppose

$$\max\Big[G_b(T^{n-1}p, T^np, T^np), G_b(T^np, T^{n+1}p, T^{n+1}p), \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}\Big] = \frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}.$$

Then, we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq k\frac{G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)}{2}. \tag{12}$$

By property (v) of G_b-metric space, we have

$$G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) \leq s\Big[G_b(T^{n-1}p, T^np, T^np) + G_b(T^np, T^{n+1}p, T^{n+1}p)\Big]. \tag{13}$$

Using inequality (13) in (12), we get

$$G_b(T^np, T^{n+1}p, T^{n+1}p) \leq sk\Big[\frac{[G_b(T^{n-1}p, T^np, T^np) + G_b(T^np, T^{n+1}p, T^{n+1}p)]}{2}\Big].$$

It gives that

$$\begin{aligned} G_b(T^np, T^{n+1}p, T^{n+1}p) &\leq \frac{sk}{2-sk}G_b(T^{n-1}p, T^np, T^np) \\ &= \lambda G_b(T^{n-1}p, T^np, T^np) \leq \cdots \leq \lambda^nG_b(p, Tp, Tp), \end{aligned}$$

where $\lambda = \frac{sk}{2-sk}$. Since $k, \lambda < 1$, so as $n \to \infty$, we get $G_b(p, Tp, Tp) = 0$ and hence in all cases $p = Tp$ i.e. $p \in F(T)$. Hence T has property P.
Theorem 5 Under the contraction of theorem 2, T has property P.
Proof. By theorem 2, T has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix

$n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $P \neq Tp$.

$$\begin{aligned}
G_b(p, Tp, Tp &= G_b(T^n p, T^{n+1} p, T^{n+1} p) \\
&\leq k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^{n-1}p, T^n p, T^n p), \\
&\quad G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p), G_b(T^n p, T^n p, T^n p), \\
&\quad G_b(T^n p, T^{n+1}p, T^{n+1}p)\Big] \\
&\leq k \max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), \\
&\quad G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big].
\end{aligned}$$

Here we have three cases,
Case (i)Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big] = G_b(T^{n-1}p, T^n p, T^n p).$$

Then we get,

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq kG_b(T^{n-1}p, T^n p, T^n p) \leq \cdots \leq k^n G_b(p, Tp, Tp).$$

Case (ii)Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big] = G_b(T^n p, T^{n+1}p, T^{n+1}p).$$

Then we get,

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq kG_b(T^n p, T^{n+1}p, T^{n+1}p),$$

which is contradiction, since $k < 1$.
Case (iii)Suppose

$$\max\Big[G_b(T^{n-1}p, T^n p, T^n p), G_b(T^n p, T^{n+1}p, T^{n+1}p), G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p)\Big] = G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p).$$

Then we get,

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq kG_b(T^{n-1}p, T^{n+1}p, T^{n+1}p). \tag{14}$$

By property (v) of G_b-metric space, we have

$$G_b(T^{n-1}p, T^{n+1}p, T^{n+1}p) \leq s\Big[G_b(T^{n-1}p, T^n p, T^n p) + G_b(T^n p, T^{n+1}p, T^{n+1}p)\Big]. \tag{15}$$

Using inequality (15) in (14), we get

$$G_b(T^n p, T^{n+1}p, T^{n+1}p) \leq sk\Big[G_b(T^{n-1}p, T^n p, T^n p) + G_b(T^n p, T^{n+1}p, T^{n+1}p)\Big].$$

It gives that

$$\begin{aligned}
G_b(T^n p, T^{n+1}p, T^{n+1}p) &\leq \frac{sk}{1 - sk} G_b(T^{n-1}p, T^n p, T^n p) \\
&= \lambda G_b(T^{n-1}p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp),
\end{aligned}$$

where $\lambda = \frac{sk}{1-sk}$. Since $k, \lambda < 1$, so as $n \to \infty$, we get $G_b(p, Tp, Tp) = 0$ and hence in all cases $p = Tp$ i.e. $p \in F(T)$. Hence T has property P.

Theorem 6 Under the contraction of theorem 3, T has property P.

Proof. By theorem 3, T has a fixed point. Therefore $F(T^n) \neq \emptyset$, each $n \in \mathbb{N}$. Fix $n > 1$ and assume that $p \in F(T^n)$. To show that $p \in F(T)$. Suppose that $P \neq Tp$.

$$\begin{aligned} G_b(p, Tp, Tp) &= G_b(T^n p, T^{n+1} p, T^{n+1} p) \\ &\leq \Big[\alpha G_b(T^{n-1} p, T^n p, T^n p) + \beta G_b(T^{n-1} p, T^n p, T^n p) \\ &\quad + \gamma G_b(T^n p, T^{n+1} p, T^{n+1} p) + \delta G_b(T^n p, T^{n+1} p, T^{n+1} p)\Big] \\ &\leq (\alpha + \beta) G_b(T^{n-1} p, T^n p, T^n p) + (\gamma + \delta) G_b(T^n p, T^{n+1} p, T^{n+1} p). \end{aligned}$$

It gives that

$$\begin{aligned} G_b(p, Tp, Tp) &\leq \frac{(\alpha + \beta)}{1 - (\gamma + \delta)} G_b(T^{n-1} p, T^n p, T^n p) \\ &= \lambda G_b(T^{n-1} p, T^n p, T^n p) \leq \cdots \leq \lambda^n G_b(p, Tp, Tp), \end{aligned}$$

where $\lambda = \frac{(\alpha+\beta)}{1-(\gamma+\delta)}$. Since $\lambda < 1$, so as $n \to \infty$, we get $G_b(p, Tp, Tp) = 0$ and hence $p = Tp$ i.e. $p \in F(T)$. Hence T has property P.

Example 3 Let us define $G_b(x, y, z) = |x-y| + |y-z| + |x-z|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$. Then

(i)

$$\begin{aligned} &G_b(T(x), T(y), T(z)) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}| \\ &= \frac{1}{3}[|x-y| + |y-z| + |x-z|] \\ &\leq k \max \Big[G_b(x, y, z), G_b(x, T(x), T(x)), G_b(y, T(y), T(y)), G_b(z, T(z), T(z)), \\ &\quad \frac{[G_b(x, T(y), T(y)) + G_b(z, T(x), T(x))]}{2}, \frac{[G_b(x, T(y), T(y)) + G_b(y, T(x), T(x))]}{2}, \\ &\quad \frac{[G_b(y, T(z), T(z)) + G_b(z, T(y), T(y))]}{2}, \frac{[G_b(x, T(z), T(z)) + G_b(z, T(x), T(x))]}{2}\Big]. \end{aligned}$$

(ii)

$$\begin{aligned} &G_b(T(x), T(y), T(z)) = |\frac{x}{3} - \frac{y}{3}| + |\frac{y}{3} - \frac{z}{3}| + |\frac{x}{3} - \frac{z}{3}| \\ &= \frac{1}{3}[|x-y| + |y-z| + |x-z|] \\ &\leq k \max \Big[G_b(x, y, z), G_b(x, T(x), T(x)), G_b(y, T(y), T(y)), G_b(x, T(y), T(y)), \\ &\quad G_b(y, T(x), T(x)), G_b(z, T(z), T(z)), \Big]. \end{aligned}$$

(iii)

$$
\begin{aligned}
&G_b(T(x),T(y),T(z)) = |\frac{x}{3}-\frac{y}{3}|+|\frac{y}{3}-\frac{z}{3}|+|\frac{x}{3}-\frac{z}{3}|\\
&= \frac{1}{3}[|x-y|+|y-z|+|x-z|]\\
&= \frac{1}{9}[|x-y|+|y-z|+|x-z|]+\frac{2}{9}[|x-y|+|y-z|+|x-z|]\\
&\le \frac{1}{9}\Big[|x-y|+|y-z|+|x-z|\Big]+\frac{2}{9}\Big[|x|+|y|+3|z|\Big]\\
&\le \frac{1}{9}\Big[|x-y|+|y-z|+|x-z|\Big]+\frac{1}{3}\Big[|\frac{2x}{3}|+|\frac{2y}{3}|]+\frac{1}{3}|6z|\Big]\\
&\le \frac{1}{9}\Big[|x-y|+|y-z|+|x-z|\Big]+\frac{1}{3}\Big[|x-\frac{x}{3}|+|y-\frac{y}{3}|\Big]+\frac{1}{3}\Big[9|z-\frac{z}{3}\Big]\\
&\le \frac{1}{9}\Big[|x-y|+|y-z|+|x-z|\Big]+\frac{1}{6}\Big[2|x-\frac{x}{3}|\Big]+\frac{1}{6}\Big[2|y-\frac{y}{3}|\Big]+\frac{1}{2}\Big[2|z-\frac{z}{3}\Big]\\
&\le \Big[\alpha G_b(x,y,z)+\beta G_b(x,T(x),T(x))+\gamma G_b(y,T(y),T(y))+\delta G_b(z,T(z),T(z))\Big],
\end{aligned}
$$

where $\alpha=\frac{1}{9},\beta=\frac{1}{6},\gamma=\frac{1}{6},\delta=\frac{1}{2}$ and $\alpha+\beta+\gamma+\delta=\frac{17}{18}<1$.

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