Fixed point theorems satisfying Φ- maps in G_b-metric space

D. R. Nhavi[a] and C. T. Aage[b]

[a] and [b] Department of Mathematics, School of Mathematical Sciences, Kavayitri Bahinabai Chaudhari North Maharashtra University, Jalgaon, India-425001

E-mail: dnyaneshwar.nhavi@gmail.com

E-mail: caage17@gmail.com

FIXED POINT THEOREMS SATISFYING Φ- MAPS IN G_b-METRIC SPACE

D. R. NHAVI AND C. T. AAGE

Abstract. In this paper, we prove fixed point theorems for self mapping $T : X \to X$ in a complete G_b-metric space for a Φ-maps as $\phi : [0, +\infty] \to [0, +\infty]$ be a nondecreasing map with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t\epsilon(0, +\infty)$ and also prove uniqueness for such fixed points in respective contractions. Our results are supported by an example.

1. Introduction and Preliminaries

The fixed point theory which plays very important role in applied mathematics and sciences. So the metric spaces are generalized by many authors by various ways. Czerwik [6] introduced b-metric space. Zead Mustafa and Brailey Sims [11] coined the concept of G-metric space. A. Aghajani, M. Abbas and J. R. Roshan [2] extended the G-metric space with b-metric space and develop the new structure of metric space, which is generalized metric space called G_b-metric space. In this paper for a self mapping in a G_b metric space we study some fixed point theorems under some contractions [15], [10]-[9] related to a nondecreasing map [4]$\phi : [0, +\infty] \to [0, +\infty]$ with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t\epsilon(0, +\infty)$.

2. Basic Concepts

A b-metric space is defined by Czerwik [6] as follows.

Definition 2.1. [6] Let X be a non empty set and the mapping $d : X \times X \to [0, \infty)$. The mapping d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

Key words and phrases. G-metric spaces; b-metric spaces; G_b-metric spaces; contraction mappings.

(iii) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then d is called a b-metric on X. The ordered pair (X, d) is called b-metric space with coefficient s.

Definition 2.2. [11] Let X be a non empty set and the mapping $G : X \times X \times X \to [0, \infty)$. The mapping G satisfies

(i) $G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$,
(ii) $0 < G(x, x, y)$ for all $x, y \in X$,
(iii) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(iv) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality). Then G is called a G-metric on X. And (X, G) is called G-metric space.

Aghajani and et.al [2] defined G_b- metric space as follows

Definition 2.3. [2] Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G_b : X \times X \times X \to R^+$ satisfies:

(i) $G_b(x, y, z) = 0$ if $x = y = z$ for all $x, y, z \in X$,
(ii) $0 < G_b(x, x, y)$ for all $x, y, z, \epsilon X$ with $x \neq y$,
(iii) $G_b(x, x, y) \leq G_b(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(iv) $G_b(x, y, z) = G_b(px, z, y)$, where p is a permutation of x, y, z (symmetry),
(v) $G_b(x, y, z) \leq s[G_b(x, a, a) + G_b(a, y, z)]$.

Then G_b is called a generalized b-metric or G_b-metric on X. The ordered pair (X, G_b) is called generalized b-metric or G_b-metric space.

Following example shows that a G_b-metric on X need not be a G-metric on X.

Example 2.4. [2] Let (X, G) be a G-metric space and $G_*(x, y, z) = G(x, y, z)^p$; where $p > 1$ is a real number. Note that G_* is a G_b-metric with $s = 2^{P-1}$. Obviously, G_* satisfies conditions (i) to (iv) of the G_b-metric space, so it suffices to show that condition (v) of G_b-metric space is hold. If $1 < p < \infty$, then the convexity of the function $f(x) = x^p (x > 0)$ implies that $(a + b)^p \leq 2^{p-1}(a^p + b^p)$. Thus for each $x, y, z, a \in X$ we obtain

$$\begin{aligned} G_*(x, y, z) = G(x, y, z)^p &\leq (G(x, a, a) + G(a, y, z))^p \\ &\leq 2^{p-1}(G(x, a, a)^p + G(a, y, z)^p) \\ &= 2^{p-1}(G_*(x, a, a) + G_*(a, y, z)). \end{aligned}$$

So G_* is a G_b-metric with $s = 2^{p-1}$.

Also in the above example, (X, G_*) is not necessarily a G-metric space.

Example 2.5. Let $X = R$ and let

$$G_b(x, y, z) = \max\{|x - y|^2, |y - z|^2, |x - z|^2\}.$$

Then (X, G_b) is a G_b-metric space with the coefficient $s = 2$.

Example 2.6. Let $X = R^+,\ p > 1$ a constant and $G_b : X \times X \times X \to R^+$ be defined by

$$G_b(x, y, z) = \max(2x, y, z))^p - |4x - y - z|^p,$$

for all $x, y, z \in X$. Then (X, G_b) is a G_b-metric space with $s > 1$.

Definition 2.7. [2] Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be :

(i) G_b-Cauchy sequence if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \epsilon$;

(ii) G_b-convergent to a point $x \in X$ if, for each $\epsilon > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that, for all $m, n \geq n_0$, $G(x_n, x_m, x) < \epsilon$.

Proposition 2.8. [2] *Let (X, G_b) be a G_b-metric space. Then the following are equivalents:*

(i) *$\{x_n\}$ is G_b-convergent to x.*

(ii) *$G_b(x_n, x_n, x) \to 0$, as $n \to \infty$.*

(iii) *$G_b(x_n, x, x) \to 0$, as $n \to \infty$.*

Proposition 2.9. [2] *Let (X, G_b) be a G_b-metric space. Then the following are equivalents:*

(i) *The sequence $\{x_n\}$ is G_b-Cauchy.*

(ii) *For every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G_b(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq n_0$.*

Definition 2.10. [2] A G_b-metric space X is called G_b-complete if every G_b-Cauchy sequence is G_b-convergent in X.

3. Main results

Our first main result is

Definition 3.1. [4] Let Φ be the set all functions ϕ such that $\phi : [0, +\infty] \to [0, +\infty]$ be a nondecreasing function with

(i) $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$,

(ii) $\phi(t) < t$ for all $t \in (0, +\infty)$,

(iii) $\phi(0) = 0$.

Then $\phi \in \Phi$, ϕ is called Φ-maps.

Theorem 3.2. *Let* (X, G_b) *be a complete* G_b*-metric space with and let* $T : X \to X$ *be a mapping satisfying*

$$G_b(Tx, Ty, Tz) \leq k\phi(G_b(x, y, z)) \tag{3.1}$$

for all $x, y, z \in X,\ \phi \in \Phi,\ sk \in [0, 1)$. *Then* T *has a unique fixed point (say* p*, i.e.,* $Tp = p$*), and* T *is* G_b*-continuous at* p.

Proof: Let $x_0 \in X$ and the mapping $T : X \to X$ be a self map. Then, we get a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$. If $x_n = x_{n-1}$ for each $n \in \mathbb{N}$. Then clearly $\{x_n\}$ is G_b-Cauchy sequence. Suppose $x_n \neq x_{n-1}$ for each $n \in \mathbb{N}$. We claim that $\{x_n\}$ is a G_b -Cauchy sequence in X, for $n \in \mathbb{N}$. Consider for $n \in \mathbb{N}$,

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &= G_b(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq k\phi(G_b(x_{n-1}, x_n, x_n)) \\ &\leq k^2\phi^2(G_b(x_{n-2}, x_{n-1}, x_{n-1})) \\ &\leq k^3\phi^3(G_b(x_{n-3}, x_{n-2}, x_{n-2})) \cdots \leq k^n\phi^n(G_b(x_0, x_1, x_1)). \end{aligned}$$

For given $\epsilon > 0$, since $\lim_{n\to\infty} \phi^n(G_b(x_0, x_1, x_1)) = 0$ and $\phi(\epsilon) < \epsilon$ there is an integer n_0 such that

$$\phi^n(G_b(x_0, x_1, x_1)) < \frac{\epsilon}{s} - k\phi(\epsilon),\ sk \in [0, 1),\ n \geq n_0. \tag{3.2}$$

Hence

$$G_b(x_n, x_{n+1}, x_{n+1}) < \frac{\epsilon}{s} - k\phi(\epsilon),\ sk \in [0, 1),\ n \geq n_0. \tag{3.3}$$

For $n, m \in N,\ n < m$, we claim that

$$G_b(x_n, x_m, x_m) < \epsilon, \tag{3.4}$$

for all $m, n \geq n_0$. We prove inequality (3.4) by induction on m, by equation (3.3) the inequality (3.4) hold for $m = n+1$. Assume that inequality (3.4) hold for $m = k$, therefore $G_b(x_n, x_k, x_k) <$

ϵ . Consider $m = k+1$,

$$\begin{aligned} G_b(x_n, x_m, x_m) &= G_b(x_n, x_{k+1}, x_{k+1}) \\ &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, x_{k+1}, x_{k+1})] \\ &= s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(Tx_n, Tx_k, Tx_k))] \\ &\leq s\big[G_b(x_n, x_{n+1}, x_{n+1}) + \phi(G_b(x_n, x_k, x_k))\big] \\ &< s[\frac{\epsilon}{s} - k\phi(\epsilon) + k\phi(\epsilon)] \\ &= \epsilon. \end{aligned}$$

Therefore, by induction on m the inequality (3.4) hold for all $n \geq m \geq n_0$. Hence $\{x_n\}$ is a G_b-Cauchy sequence in X. By G_b-completeness of X, there exists $p \in X$ such that $\{x_n\}$ is G_b converges to p. Now we show that p is fixed point of T. Suppose that $T(p) \neq p$.

$$\begin{aligned} G_b(x_n, Tp, Tp) &\leq s\big[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tp, Tp)\big] \\ &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + k\phi(G_b(x_n, p, p))] \\ &< s[G_b(x_n, x_{n+1}, x_{n+1}) + kG_b(x_n, p, p)]. \end{aligned}$$

As $n \to +\infty, x_n \to p$

$G_b(p, Tp, Tp) \leq 0$ and since $G_b(p, Tp, Tp) \geq 0$. Then $Tp = p$.

This is contradiction to $Tp \neq p$. Therefore p is a fixed point of T. For uniqueness suppose $q \neq p$ and q is another fixed point of T, $Tq = q$.

$$\begin{aligned} G_b(x_n, Tq, Tq) &\leq s[G_b(x_n, x_{n+1}, x_{n+1}) + G_b(x_{n+1}, Tq, Tq)] \\ &\leq [G_b(x_n, x_{n+1}, x_{n+1}) + k\phi(G_b(x_n, q, q)] \\ &< s[G_b(x_n, x_{n+1}, x_{n+1}) + kG_b(x_n, q, q)]. \end{aligned}$$

As $n \to \infty, x_n \to p$ and $Tq = q$, we get

$$\begin{aligned} G_b(p, q, q) &\leq s[G_b(p, p, p) + kG_b(p, q, q)] \\ &= skG_b(p, q, q). \end{aligned}$$

It follows that,

$$(1 - sk)G_b(p, q, q) < 0$$

$G_b(p,q,q) = 0$; since $sk \in [0,1)$. To show that T is G_b-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} y_n = p$. Consider

$$\begin{aligned} G_b(p, T(y_n), T(y_n)) &\leq G_b(Tp, T(y_n), T(y_n) \\ &\leq k\phi(G_b(p, y_n, y_n) \\ &< kG_b(p, y_n, y_n). \end{aligned}$$

As As $n \to \infty, y_n \to p$, we get

$$\begin{aligned} &G_b(p, T(y_n), T(y_n) < kG_b(p,p,p) \\ &G_b(p, T(y_n), T(y_n) = 0. \end{aligned}$$

Thus

$$T(y_n) = p = Tp.$$

It is proved that T is G_b-continuous at p.

Corollary 3.3. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying for some $m \in N$*

$$G_b(T^m(x), T^m(y), T^m(z)) \leq k\phi(G_b(x,y,z)); \tag{3.5}$$

for all $x, y, z \in X,\ sk \in [0,1)$. Then T has a unique fixed point (say u, i.e., $Tu = u$),and T^m is G_b-continuous at p.

Proof: Here $T(u) = T(T^m u) = T^{m+1}u = T^m(Tu)$. Therefore by Theorem (3.2) we conclude that T^m has a fixed point say p. Also we have Tu a fixed point to T^m. So $Tu = u$, and T has unique fixed point.

Corollary 3.4. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying for some $m \in N$*

$$G_b(Tx, Ty, Ty) \leq k\phi(G_b(x,y,y)); \tag{3.6}$$

for all $x, y, z \in X, sk \in [0,1)$. Then T has a unique fixed point (say u, i.e., $Tu = u$),and T is G_b-continuous at p.

Proof: Taking $z = y$ in Theorem (3.2).

Corollary 3.5. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying for some $m \in N$*

$$G_b(Tx, Ty, Tz) \leq kG_b(x,y,z); \tag{3.7}$$

for all $x, y, z \in X$ $k \in [0, 1)$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G_b-continuous at p.

Proof: To prove this corollary we define the ϕ function as $\phi : [0, +\infty] \to [0, +\infty]$ be a nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$ and $\phi(\omega) = \omega$. Clearly ϕ is nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$.
Since $G_b(Tx, Ty, Tz) \le k\phi(G_b(x, y, z))$; for all $x, y, z \in X, sk \in [0, 1)$. Therefore by Theorem (3.2) we get required result.

Corollary 3.6. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying for some $m \in N$*

$$G_b(Tx, Ty, Tz) \le \frac{G_b(x, y, z)}{1 + G_b(x, y, z)}; \tag{3.8}$$

for all $x, y, z \in X$. Then T has a unique fixed point (say u, i.e., $Tu = u$),and T is G_b-continuous at p.

Proof: To prove this corollary we define the ϕ function as $\phi : [0, +\infty] \to [0, +\infty]$ be a nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$ and $\phi(\omega) = \frac{k\omega}{1+k\omega}$. Clearly ϕ is nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$.
Since $G_b(Tx, Ty, Tz) \le k\phi(G_b(x, y, z))$; for all $x, y, z \in X$. Therefore by Theorem (3.2) we get required result.

Theorem 3.7. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying*

$$G_b(Tx, Ty, Tz) \le k\phi\Big(\max\big[G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), G_b(z, Tz, Tz)\big]\Big); \tag{3.9}$$

for all $x, y, z \in X, sk \in [0, 1)$. Then T has a unique fixed point (say p, i.e., $Tp = p$),and T is G_b-continuous at p.

Proof: Let $x_0 \in X$ and $\{x_n\}$ be a sequence in X and $x_n = Tx_{n-1} = T^n x_0$. Assume that $x_n = x_{n-1}$ for each $n \in N$. Then clearly $\{x_n\}$ is G_b-Cauchy sequence. Suppose $x_n \ne x_{n-1}$ for each $n \in \mathbb{N}$. We claim that $\{x_n\}$ is a G_b -Cauchy sequence in X, for $n \in \mathbb{N}$. Consider

$$\begin{aligned} G_b(x_n, x_{n+1}, x_{n+1}) &= G_b(Tx_{n-1}, Tx_n, Tx_n) \\ &\le k\phi\Big(\max\big[G_b(x_{n-1}, x_n, x_n), G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_n, x_{n+1}, x_{n+1})\big]\Big) \\ &\le k\phi\big(\max\big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1})\big]\big). \end{aligned}$$

case i) If $\max\big[G_b(x_{n-1}, x_n, x_n), G_b(x_n, x_{n+1}, x_{n+1})\big] = G_b(x_n, x_{n+1}, x_{n+1})$.
Then $G_b(x_n, x_{n+1}, x_{n+1}) \le k\phi(G_b(x_n, x_{n+1}, x_{n+1})) = G_b(x_n, x_{n+1}, x_{n+1})$, which is impossible.

case ii) max $\big[G_b(x_{n-1},x_n,x_n),G_b(x_n,x_{n+1},x_{n+1})\big] = G_b(x_{n-1},x_n,x_n)$, and hence

$$\begin{aligned}G_b(x_n,x_{n+1},x_{n+1}) &\le k\phi(G_b(x_{n-1},x_n,x_n))\\ &\le k^2\phi^2(G_b(x_{n-2},x_{n-1},x_{n-1}))\\ &\le k^3\phi^3(G_b(x_{n-3},x_{n-2},x_{n-2}))\cdots \le k^n\phi^n(G_b(x_0,x_1,x_1)).\end{aligned}$$

By similar way from the proof of theorem (3.2); we can show that the sequence $\{x_n\}$ is a Cauchy sequence in X. By completeness, there exists $p \in X$ such that $\{x_n\}$ is G_b converges to p. Now we show that p is fixed point of T. Suppose that $Tp \neq p$.

$$\begin{aligned}G_b(x_n,Tp,Tp) &\le s\Big[G_b(x_n,x_{n+1},x_{n+1}) + G_b(x_{n+1},Tp,Tp)\Big]\\ &\le s\Big[G_b(x_n,x_{n+1},x_{n+1}) + k\phi\Big(max\big[(G_b(x_n,p,p)),G_b(x_n,x_{n+1},x_{n+1}),\\ &\qquad G_b(p,Tp,Tp),G_b(p,Tp,Tp)\big]\Big)\Big]\\ &\le s\Big[G_b(x_n,x_{n+1},x_{n+1}) + k\phi\Big(max\big[(G_b(x_n,p,p)),G_b(x_n,x_{n+1},x_{n+1}),\\ &\qquad G_b(p,Tp,Tp)\big]\Big)\Big]. \qquad (3.10)\end{aligned}$$

case i) $max\big[(G_b(x_n,p,p)),G_b(x_n,x_{n+1},x_{n+1}),G_b(p,Tp,Tp)\big] = G_b(x_n,p,p)$, then we have

$$\begin{aligned}G_b(x_n,Tp,Tp) &\le s[G_b(x_n,x_{n+1},x_{n+1}) + k\phi G_b(x_n,p,p)]\\ &\le s[G_b(x_n,x_{n+1},x_{n+1}) + kG_b(x_n,p,p)],\end{aligned}$$

Letting $n \to \infty$, we conclude that

$$G_b(p,Tp,Tp) < 0 \;\Rightarrow p = Tp$$

. case ii) $max\big[(G_b(x_n,p,p)),G_b(x_n,x_{n+1},x_{n+1}),G_b(p,Tp,Tp)\big] = G_b(x_n,x_{n+1},x_{n+1})$ then we have

$$\begin{aligned}G_b(x_n,Tp,Tp) &\le s[G_b(x_n,x_{n+1},x_{n+1}) + k\phi G_b(x_n,x_{n+1},x_{n+1})]\\ &\le s(1+k)G_b(x_n,x_{n+1},x_{n+1}),\end{aligned}$$

Letting $n \to \infty$, we conclude that

$$\begin{aligned}&G_b(p,Tp,Tp) < s(1+k)G_b(p,p,p)\\ &G_b(p,Tp,Tp) = 0.\end{aligned}$$

It follows that

$$p = Tp$$

. case iii) $max\big[(G_b(x_n,p,p)),G_b(x_n,x_{n+1},x_{n+1}),G_b(p,Tp,Tp)\big]=G_b(p,Tp,Tp)$ then we have

$$\begin{aligned}G_b(x_n,Tp,Tp)&\leq s[G_b(x_n,x_{n+1},x_{n+1})+k\phi G_b(p,Tp,Tp)]\\&\leq s[G_b(x_n,x_{n+1},x_{n+1})+kG_b(p,Tp,Tp)],\end{aligned}$$

Letting $n\to\infty$, we conclude that

$$\begin{aligned}&G_b(p,Tp,Tp)<s[G_b(p,p,p)+kG_b(p,Tp,Tp)],\\&G_b(p,Tp,Tp)=skG_b(p,Tp,Tp).\end{aligned}$$

It follows that

$$\begin{aligned}(1-sk)G_b(p,Tp,Tp)&<0\\G_b(p,Tp,Tp)&=0;\;\; sk\in[0,1)\\p&=Tp.\end{aligned}$$

This is contradiction to $Tp\neq p$. Therefore p is a fixed point of T. For uniqueness suppose $q\neq p$ and q is another fixed point of T, and $Tq=q$.

$$\begin{aligned}G_b(x_n,Tq,Tq)&\leq s[G_b(x_n,x_{n+1},x_{n+1})+G_b(x_{n+1},Tq,Tq)]\\&<s\Big[G_b(x_n,x_{n+1},x_{n+1})+k\phi\Big(max\big[(G_b(x_n,q,q)),G_b(x_n,x_{n+1},x_{n+1})\\&\quad G_b(q,Tq,Tq),G_b(q,Tq,Tq)\big]\Big)\Big]\\&<s\Big[G_b(x_n,x_{n+1},x_{n+1})+k\phi\Big(max\big[(G_b(x_n,q,q)),G_b(x_n,x_{n+1},x_{n+1})\\&\quad G_b(q,Tq,Tq)\big]\Big)\Big].\end{aligned}$$

This is same as equation (3.10) we replace p by q in equation (3.10) and as $n\to\infty, x_n\to p$ and $Tq=q$.

$$\begin{aligned}G_b(p,q,q)&=0\\p&=q.\end{aligned}$$

To show that T is G_b-continuous at p, let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty}y_n=p$. Consider

$$\begin{aligned}G_b(p,T(y_n),T(y_n))&\leq G_b(Tp,T(y_n),T(y_n)\\&\leq k\phi\Big(max\big[G_b(p,y_n,y_n),G_b(p,Tp,Tp),G_b(y_n,T(y_n),T(y_n)),G_b(y_n,T(y_n),T(y_n))\big]\Big).\end{aligned}$$

As $n \to \infty, y_n \to p$

$$G_b(p, T(y_n), T(y_n) < kG_b(p, p, p)$$

$$G_b(p, T(y_n), T(y_n) = 0.$$

Thus

$$T(y_n) = p = Tp.$$

It is proved that T is G_b-continuous at p.

Corollary 3.8. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying for some $m \in N$*

$$G_b(Tx, Ty, Tz) \leq k \max \big[G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), G_b(z, Tz, Tz)\big]; \qquad (3.11)$$

for all $x, y, z \in X$ and $sk \in [0, 1)$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G_b-continuous at p.

Proof: To prove this corollary we define the ϕ function as $\phi : [0, +\infty] \to [0, +\infty]$ be a nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$ and $\phi(\omega) = k\omega$. Clearly ϕ is nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$.
Since $G_b(Tx, Ty, Tz) \leq k\phi\Big(max\big[G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), G_b(z, Tz, Tz)\big]\Big)$; for all $x, y, z \in X$. Therefore by Theorem (3.7) we get required result.

Corollary 3.9. *Let (X, G_b) be a complete G_b-metric space and let $T : X \to X$ be a mapping satisfying for some $m \in N$*

$$G_b(Tx, Ty, Tz) \leq \frac{M(x, y, z)}{1 + M(x, y, z)}; \qquad (3.12)$$

where $M(x, y, z) = kmax\big[G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Ty), G_b(z, Tz, Tz)\big]$; for all $x, y, z \in X$. Then T has a unique fixed point (say u, i.e., $Tu = u$),and T is G_b-continuous at p.

Proof: To prove this corollary we define the ϕ function as $\phi : [0, +\infty] \to [0, +\infty]$ be a nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$ and $\phi(\omega) = \frac{\omega}{1+\omega}$. Clearly ϕ is nondecreasing function with $\lim_{n,m\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$.
Since $G_b(Tx, Ty, Tz) \leq \phi(M(x, y, z))$; for all $x, y, z \in X$. Therefore by Theorem (3.7) we get required result.

Example 3.10. Let us define $G_b(x, y, z) = \big|x - y\big| + \big|y - z\big| + \big|x - z\big|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$ and $\phi(t) = t$ Then

(i)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{x}{3}-\frac{y}{3}\Big| + \Big|\frac{y}{3}-\frac{z}{3}\Big| + \Big|\frac{x}{3}-\frac{z}{3}\Big| \\ &= \frac{1}{3}\big[|x-y|+|y-z|+|x-z|\big] \\ &\leq \frac{1}{2}\big[|x-y|+|y-z|+|x-z|\big] \\ &= k(G_b(x,y,z)) \\ &= k\phi(G_b(x,y,z)). \end{aligned}$$

(ii)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{x}{3}-\frac{y}{3}\Big| + \Big|\frac{y}{3}-\frac{z}{3}\Big| + \Big|\frac{x}{3}-\frac{z}{3}\Big| \\ &\leq 2k\Big|(x-\frac{x}{3})\Big| \\ &= k\max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(Tx,y,z)\big] \\ &= k\phi\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big). \end{aligned}$$

Example 3.11. Let us define $G_b(x,y,z) = \big|2x-y\big| + \big|2y-z\big| + \big|2z-x\big|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$ and $\phi(t) = t$ Then

(i)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{2x}{3}-\frac{y}{3}\Big| + \Big|\frac{2y}{3}-\frac{z}{3}\Big| + \Big|\frac{2z}{3}-\frac{x}{3}\Big| \\ &\leq \frac{1}{2}\Big[\big|2x-y\big| + \big|2y-z\big| + \big|2z-x\big|\Big] \\ &= kG_b(x,y,z) \\ &= k\phi\Big(G_b(x,y,z)\Big). \end{aligned}$$

(ii)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{2x}{3}-\frac{y}{3}\Big| + \Big|\frac{2y}{3}-\frac{z}{3}\Big| + \Big|\frac{2z}{3}-\frac{x}{3}\Big| \\ &\leq \frac{1}{2}\Big[|2x-\frac{x}{3}\Big| + \Big|\frac{2x}{3}-\frac{x}{3}\Big| + \Big|\frac{2x}{3}-x|\Big] \\ &= \frac{1}{2}G_b(x,Tx,Tx) \\ &= k\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big) \\ &= k\phi\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big). \end{aligned}$$

Example 3.12. Let us define $G_b(x,y,z) = |x-y| + |y-z| + |x-z|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$ and $\phi(t) = \frac{t}{2}$ Then

(i)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{x}{3} - \frac{y}{3}\Big| + \Big|\frac{y}{3} - \frac{z}{3}\Big| + \Big|\frac{x}{3} - \frac{z}{3}\Big| \\ &= \frac{1}{3}\big[|x-y| + |y-z| + |x-z|\big] \\ &\leq \frac{2}{3} \times \frac{1}{2}\big[|x-y| + |y-z| + |x-z|\big] \\ &= k\Big[\frac{1}{2}G_b(x,y,z)\Big] \\ &= k\phi(G_b(x,y,z)). \end{aligned}$$

(ii)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{x}{3} - \frac{y}{3}\Big| + \Big|\frac{y}{3} - \frac{z}{3}\Big| + \Big|\frac{x}{3} - \frac{z}{3}\Big| \\ &\leq 2k\Big|(x - \frac{x}{3})\Big| \\ &= \frac{2}{3} \times \frac{1}{2}\Big[2\Big|x - \frac{x}{3}\Big|\Big] \\ &= k\Big[\frac{1}{2}\max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(Tx,y,z)\big]\Big] \\ &= k\phi\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big). \end{aligned}$$

Example 3.13. Let us define $G_b(x,y,z) = \big|2x-y\big| + \big|2y-z\big| + \big|2z-x\big|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$ and $\phi(t) = \frac{t}{2}$ Then

(i)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= \Big|\frac{2x}{3} - \frac{y}{3}\Big| + \Big|\frac{2y}{3} - \frac{z}{3}\Big| + \Big|\frac{2z}{3} - \frac{x}{3}\Big| \\ &= \frac{1}{3}\Big[\big|2x-y\big| + \big|2y-z\big| + \big|2z-x\big|\Big] \\ &\leq k\Big(\frac{1}{2}G_b(x,y,z)\Big) \\ &= k\phi\Big(G_b(x,y,z)\Big). \end{aligned}$$

(ii)

$$
\begin{aligned}
G_b(Tx,Ty,Tz) &= \Big|\frac{2x}{3}-\frac{y}{3}\Big| + \Big|\frac{2y}{3}-\frac{z}{3}\Big| + \Big|\frac{2z}{3}-\frac{x}{3}\Big| \\
&\le \frac{k}{2}\Big[|2x-\frac{x}{3}\Big| + \Big|\frac{2x}{3}-\frac{x}{3}\Big| + \Big|\frac{2x}{3}-x|\Big] \\
&= k\Big(\frac{1}{2}max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big) \\
&= k\phi\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big).
\end{aligned}
$$

Example 3.14. Let us define $G_b(x,y,z) = |x-y| + |y-z| + |x-z|$ and let $x \in X$. Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$ and $\phi(t) = \frac{t}{2}$ Then

(i)

$$
\begin{aligned}
G_b(Tx,Ty,Tz) &= \Big|\frac{x}{3}-\frac{y}{3}\Big| + \Big|\frac{y}{3}-\frac{z}{3}\Big| + \Big|\frac{x}{3}-\frac{z}{3}\Big| \\
&= \frac{1}{3}\big[|x-y| + |y-z| + |x-z|\big] \\
&\le \frac{2}{3} \times \frac{1}{2}\big[|x-y| + |y-z| + |x-z|\big] \\
&= k\Big[\frac{1}{2}G_b(x,y,z)\Big] \\
&= k\phi(G_b(x,y,z)).
\end{aligned}
$$

(ii)

$$
\begin{aligned}
G_b(Tx,Ty,Tz) &= \Big|\frac{x}{3}-\frac{y}{3}\Big| + \Big|\frac{y}{3}-\frac{z}{3}\Big| + \Big|\frac{x}{3}-\frac{z}{3}\Big| \\
&\le 2k\Big|(x-\frac{x}{3})\Big| \\
&= \frac{2}{3} \times \frac{1}{2}\Big[2\Big|x-\frac{x}{3}\Big|\Big] \\
&= k\Big[\frac{1}{2}\max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(Tx,y,z)\big]\Big] \\
&= k\phi\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big).
\end{aligned}
$$

Example 3.15. Let us define $G_b(x,y,z) = \big|2x-y\big| + \big|2y-z\big| + \big|2z-x\big|$ and let $x \in X$ and $G_b(x,y,z) = G(x,y,z)^p$; where $p > 1$ is a real number.Then (X, G_b) be a complete G_b-metric space. Let $T(x) = \frac{x}{3}$. Without loss of generality, we assume $x > y > z$ and $\phi(t) = \frac{t}{2}$ Then

(i)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= G(Tx,Ty,Tz)^p \\ &= \Big(\Big|\frac{2x}{3}-\frac{y}{3}\Big| + \Big|\frac{2y}{3}-\frac{z}{3}\Big| + \Big|\frac{2z}{3}-\frac{x}{3}\Big|\Big)^p \\ &= (\frac{1}{3})^p\Big[\big|2x-y\big| + \big|2y-z\big| + \big|2z-x\big|\Big]^p \\ &\leq k\Big(\frac{1}{2}G_b(x,y,z)\Big) \\ &= k\phi\Big(G_b(x,y,z)\Big). \end{aligned}$$

(ii)

$$\begin{aligned} G_b(Tx,Ty,Tz) &= G(Tx,Ty,Tz)^p \\ &= \Big(\Big|\frac{2x}{3}-\frac{y}{3}\Big| + \Big|\frac{2y}{3}-\frac{z}{3}\Big| + \Big|\frac{2z}{3}-\frac{x}{3}\Big|\Big)^p \\ &\leq k\Big(\frac{1}{2}\Big[|2x-\frac{x}{3}\Big| + \Big|\frac{2x}{3}-\frac{x}{3}\Big| + \Big|\frac{2x}{3}-x|\Big]^p\Big) \\ &= k\Big(\frac{1}{2}max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]^p\Big) \\ &= k\phi\Big(max\big[G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Ty), G_b(z,Tz,Tz)\big]\Big). \end{aligned}$$

References

[1] A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, in press.

[2] A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, Filomat, in press.

[3] I. A. Bakhtin, The contraction mapping principle in quasi-metric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30(1989),26-37.

[4] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proceedings of the American Mathematical Society, vol. 62, no. 2, pp. 344348, 1977.

[5] M. A. Khamsi, W. A. Kirk, An Introduction to Metric Spaces and Fixed Point Theory,A Wiley-lnterscience Publication.

[6] S. Czerwik, Contraction mappings in b-metric space, Acta Mat. Inf. Univ.Ostraviensis 1(1993), 5-11.

[7] W. A. Kirk and B. Sims, Handbook of metric fixed point theory, Kluwer, Academic Publishers, 0Dordrecht, 2001.

[8] W. Shatanawi, Hindawi Publishing Corporation, Fixed Point Theory and Applications, Volume 2010, Article ID 181650,9 pages doi:10.1155/2010/181650

[9] W. Shatanawi, Some fixed point theorems in ordered G-metric spaces and applications. Abstr. Appl. Anal. 2011, Article ID 126205 (2011).

[10] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. Thesis, The University of Newcastle, Callaghan, Australia, 2005.

[11] Z. Mustafa and B. Sims, A new approach to generalized metric spaces Journal of Nonlinear and Convex Analysis, vol.7, no.2, pp. 289- 297, 2006.

[12] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete G-metric spaces. Fixed Point Theory Appl. 2009, Article ID 917175 (2009).

[13] Z. Mustafa and B. Sims, Some remarks concerning D-metric spaces, in: Proceedings of the International Conference on Fixed Point Theory and Applications, Yokohama, Japan, 2004, pp. 189198.

[14] Z. Mustafa and H. Obiedat, A fixed point theorem of Reich in G-metric spaces. CUBO 12(1), 83-93 (2010).

[15] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete Gmetric spaces, Fixed Point Theory and Applications, vol. 2008, Article ID 189870, 12 pages, 2008.

[16] Z. Mustafa, J.R. Roshan and V. Parvaneh, Coupled coincidence point results for (ψ, ϕ)-weakly contractive mappings in partially ordered Gb-metric spaces, Fixed Point Theory Appl. 2013:206 (2013).

[17] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in G-metric spaces. Int. J. Math. Math. Sci. 2009, Article ID 283028 (2009).

D. R. Nhavi Department of Mathematics

E-mail address: dnyaneshwar.nhavi@gmail.com

C. T. Aage Department of Mathematics North Maharashtra University, Jalgaon, India

E-mail address: caage17@gmail.com